Macroeconomics I: Economic Growth

Problem Set 1: The Solow-Swan model

Solutions

The following provides only a possible solution to the problem set.

General remark: We use the terms "Balanced Growth Path" and "Steady State" interchangeably. On a BGP or SS, all variables grow with a constant rate. This rate can take any real value (positive, negative or zero). This definition is slightly different to the one given by BSiM in 1.2.4 in which SS is a particular case of BGP where the growth rate is zero.

1 Cobb-Douglas Technology

1. In the Solow model, the saving rate \( s \in (0, 1) \) is constant and exogenously given so that

\[
I_t = sY_t = Y_t - C_t, \forall t. \tag{1}
\]

The objective of the Ramsey model studied in the next problem set will be to endogenise \( s \), i.e. to make it dependent on the state of the economy. With investment given by (1), the capital stock evolves according to

\[
\dot{K}_t = I_t - \delta K_t = sY_t - \delta K_t, \tag{2}
\]

where \( \dot{x} \) denotes the change of a variable \( x \) over time (or \( dx/dt \)) and \( \delta \) is the rate of capital depreciation assumed to be constant. Now note that \( k_t = K_t/X_tN_t \) is the capital stock per worker in efficiency units at time \( t \) and using (2) we have

\[
\dot{k}_t = \frac{d}{dt} \left( \frac{K_t}{X_tN_t} \right) = \frac{\dot{K}_t}{X_tN_t} - \frac{K_t(\dot{X}_tN_t + N_t\dot{X}_t)}{(X_tN_t)^2}
\]

\[
= \frac{\dot{K}_t}{X_tN_t} - \frac{K_t(n + \gamma)}{X_tN_t}
\]

\[
= \frac{sY_t - \delta K_t}{X_tN_t} - \frac{K_t(n + \gamma)}{X_tN_t}
\]

\[
= sf(k_t) - (n + \gamma + \delta)k_t
\]

where the second row uses the definitions of the growth rates of the population resp. technology \( n = \dot{N}/N \) and \( \gamma = \dot{X}/X \). Note that this derivation relies only on constant returns.
to scale\(^1\), not on the production function being Cobb-Douglas. The term \((n + \gamma + \delta)\) is an effective depreciation rate of capital, with \(\delta\) physical depreciation, and \(n\) and \(\gamma\) accounting for the dilution of effective capital per worker due to population growth and technological progress, respectively, i.e. \(nk\) units have to be invested to keep capital per capita constant and \(\gamma k\) units have to be invested to keep capital in efficiency unit constant. The variable is used only for derivations of the steady state and has no empirical counterpart.

The production function \(f(k) = Ak^\alpha\) is the intensive form for \(F(K, XN) = AK^\alpha (XN)^{1-\alpha}\). Technological change enters in labour augmenting (Harrod neutral) form. With constant returns to scale, \(Y_t = F(K_t, X_tN_t) = X_tN_t F\left(\frac{K_t}{X_tN_t}, 1\right) = X_tN_t F(k_t, 1) = X_tN_t f(k_t)\). (3)

Defining \(f(k_t) \equiv F(k_t, 1)\), we obtain \(y_t = Y_t/X_tN_t = f(k_t)\).

Now use (3) to obtain expressions for the marginal products of capital (MPK) and of labor (MPL). In a competitive environment, the firm will be ready to pay wages \(w_t\) up to the point where they equate the marginal product of labor. Similarly, firms are ready to pay to use capital up to the point where the marginal product of capital (it is decreasing in capital) reaches the user cost of capital.

\[
\begin{align*}
MPK &= \frac{\partial Y}{\partial K} = XN \frac{\partial F\left(\frac{K}{XN}, 1\right)}{\partial K} = XN \cdot f'(k) \cdot \left(\frac{1}{XN}\right) = f'(k) = \alpha f(k_t)/k_t \quad (4) \\
MPL &= \frac{\partial Y}{\partial N} = \frac{\partial}{\partial N} \left[ XN F\left(\frac{K}{XN}, 1\right) \right] \\
&= X \cdot f(k) + XN \cdot f'(k) \cdot \left[-K/XN^2\right] = X[f(k) - f'(k)k] = (1-\alpha) X_tF(k_t)\quad (5)
\end{align*}
\]

where the last equality employs the fact that the production function is a Cobb-Douglas function.

2. Using \(y = f(k) = Ak^\alpha\), the state variable evolves according to

\[
\dot{k} = sAk^\alpha - (n + \gamma + \delta)k. \quad (6)
\]

Define \(z = k^{1-\alpha} :\iff k = z^{1/(1-\alpha)}\). This variable transformation is useful in order to transform the (non-solvable) non-linear differential equation in capital into a (solvable) linear differential equation.

\[
\begin{align*}
\dot{k} &= \frac{1}{1-\alpha} z^{\alpha/(1-\alpha)} \dot{z} \\
\dot{z} &= sAz^{\alpha/(1-\alpha)} - (n + \gamma + \delta)z^{1/(1-\alpha)} \quad (7) \\
\ddot{z} &= (1-\alpha)z - (1-\alpha)(n + \gamma + \delta)z \\
\end{align*}
\]

\(^1\)CRS implies \(Y_t = F(K_t, X_tN_t) = \frac{Y_t}{X_tN_t} = F\left(\frac{K_t}{X_tN_t}, 1\right), y_t = f(k_t)\).
which is an ordinary autonomous linear first order differential equation with constant coefficients (see Simon and Blume or BSiM A.1). To save on notation, define \[ \zeta = (1 - \alpha)(n + \gamma + \delta) > 0, \] leading to \[ \dot{z} + \zeta z = (1 - \alpha)sA. \] (9)

Use \( e^{\zeta t} \) as an integrating factor to obtain

\[ \dot{z} e^{\zeta t} + \zeta z e^{\zeta t} = (1 - \alpha)sA e^{\zeta t}. \] (10)

The left hand side equals \( d(z e^{\zeta t})/dt \) so that integrating both side with respect to time gives

\[ e^{\zeta t} z_t + b_0 = \frac{e^{\zeta t}(1 - \alpha)sA}{\zeta} + b_1 \] (11)

where \( b_0 \) and \( b_1 \) are arbitrary constants of integration. Solving for \( z_t \) and dividing by \( e^{\zeta t} \) yields

\[ z_t = \frac{(1 - \alpha)sA}{\zeta} + \Omega e^{-\zeta t} \] (12)

where \( \Omega = (b_1 - b_0) \) is a constant of integration. It can be determined using the initial condition \( z_0 \) \( (t = 0) \):

\[ \Omega = z_0 - \frac{(1 - \alpha)sA}{\zeta}. \] (13)

Substituting back in for \( z, \zeta \), and \( \Omega \), we can retrieve the time path of \( k_t \) as a function of time and parameters only:

\[ k_1^{1-\alpha} = \frac{sA}{n + \gamma + \delta} + \left[k_0^{1-\alpha} - \frac{sA}{n + \gamma + \delta}\right] e^{-(1 - \alpha)(n + \gamma + \delta)t}. \] (14)

It is easy to see that as \( t \to \infty \), \( k_t \) approaches its steady-state value of

\[ k_{SS} = \left[\frac{sA}{n + \gamma + \delta}\right]^{1/(1-\alpha)}. \] (15)

Which is what you would also obtain by setting (6) to zero and solving for \( k \). The term in square brackets in (14) then gives the gap between \( k_{SS}^{1-\alpha} \) and \( k_1^{1-\alpha} \) at \( t = 0 \). This gap is closed proportionally at the rate \((1 - \alpha)(n + \gamma + \delta)\). As a consequence, growth of \( k \) slows down as the gap diminishes and the economy approaches its steady state.

3. Write \( g_x \) for the growth rate \( \dot{x}/x \) of a variable \( x \). From the production function, it is clear that \( g_y = \alpha g_k \), as \( \dot{y}/y = A\alpha k^{\alpha-1} \dot{k}/(A k^\alpha) \). Thus,

\[ g_y = \alpha \frac{\dot{k}}{k} = \alpha \left[sAk^{\alpha-1} - (n + \gamma + \delta)\right] = \alpha \left[sA \frac{1}{\alpha} y^{\frac{\alpha-1}{\alpha}} - (n + \gamma + \delta)\right]. \] (16)
We know from the solution for $k_{SS}$ that $y_{SS} = A\left[\frac{sA}{n+\gamma+\delta}\right]^{\frac{1-\alpha}{\alpha}}$.

Solving this for $s = (n + \gamma + \delta)A^{-\frac{1}{\alpha}}y_{SS}^{\frac{1-\alpha}{\alpha}}$ and using it in (16), we obtain

$$g_y = \alpha(n + \gamma + \delta) \left[\left(\frac{y_t}{y_{SS}}\right)^{\frac{\alpha-1}{\alpha}} - 1\right].$$

The growth rate of output if the current level $y_t$ is below the steady state value $y_{ss}$ is positive, leading to an increase in output next period. The opposite is true if $y_t > y_{ss}$. This fact documents the stability of the steady state, as output from either side converges towards the steady state.

The user cost of capital is defined by $r + \delta = f'(k)$. The rate of change of the user cost of capital is the time derivative divided by the value itself: $g_r = \frac{d(r+\delta)/dt}{r+\delta} = \frac{f''(k)\dot{k}}{f'(k)}$. Using the expression for the first and second derivative we obtain: $g_r = (\alpha - 1)\dot{k}/k$. Using the same procedure as for $y$ we obtain

$$g_r = \frac{\alpha - 1}{\alpha} g_y = (\alpha - 1) g_k$$

$$= (\alpha - 1)(n + \gamma + \delta) \left[\left(\frac{k}{k_{SS}}\right)^{\frac{\alpha-1}{\alpha}} - 1\right]$$

$$= (\alpha - 1)(n + \gamma + \delta) \left[\frac{r + \delta}{r_{SS} + \delta} - 1\right]$$

The growth rate of the net interest rate has the opposite sign to that of output or capital growth. If the economy is below it's steady state capital level $k_t < k_{SS}$ or output level $y_t < y_{ss}$, the interest rate is larger than at the steady state, $r_t > r_{SS}$ (due to decreasing returns to capital), and along transition to the steady state the growth rate of the user cost of capital is negative, as $(\alpha - 1)$ is negative.

Regarding wages, as shown in part 1: $w = X[f(k) - f'(k)k]$, so that

$$g_w = \frac{\dot{X}[f(k) - f'(k)k]}{X[f(k) - f'(k)k]} - \frac{X[f'(k)k - f''(k)k\dot{k} - f'(k)\dot{k}]}{X[f(k) - f'(k)k]}$$

$$= \frac{\dot{X}}{X} \frac{A\alpha (\alpha - 1) k^{\alpha-2}\dot{k}}{Ak^{\alpha}[1-\alpha]}$$

$$= \gamma + \alpha \dot{k}/k$$

$$= \gamma + g_y$$

The growth rate of wages is determined by the rate of technological progress (even in steady state) and on the transition path it possess higher growth rates than at steady state in the same magnitude as output in efficiency units.
To sum up: If $k_0 < k_{ss}$, the economy in efficiency units converges to the steady state thanks to positive growth rates for capital and output, the interest decreases due to decreasing marginal returns to capital, and wages grow with the rate of technological progress as well as the "catching-up" factor of output growth in efficiency units.

4. $\beta$ is the convergence rate, i.e. the proportion of the distance between $y$ and $y_{SS}$ that is closed each period. It is defined as

$$\beta = -\frac{dg_y}{d(\ln y)}.$$  \hspace{1cm} (19)

Rewriting $g_y$ as a function of $\ln y$

$$\alpha(n + \gamma + \delta) \left[ (y_{SS})^{\frac{1-\alpha}{\alpha}} e^{\frac{\alpha-1}{\alpha} \ln y} - 1 \right]$$  \hspace{1cm} (20)

and taking the derivative with respect to $\ln y$ yields

$$\beta = (1 - \alpha)(n + \gamma + \delta) \left( \frac{y}{y_{SS}} \right)^{-\frac{1-\alpha}{\alpha}},$$  \hspace{1cm} (21)

which assesses how the growth rate of output changes, if the level of output is changed proportionally.

As the value for output $y$ increases, the convergence rate declines (convergence slows down). At the steady state, $y = y_{SS}$ and $\beta = (1 - \alpha)(n + \gamma + \delta)$, which is constant, and in particular it is independent of the savings rate. Note that even at the steady state, when convergence is complete, the convergence rate $\beta$ is not zero although growth is zero. This reflects how quickly $k$ returns to the steady state, in case of a small deviation from it. You can also obtain the steady state value of $\beta$ by a first-order Taylor expansion w.r.t. $\ln k$ of the growth rate $\dot{k}/k = sAk^{\alpha-1} - (n + \gamma + \delta)$ around the steady state ($k = k_{ss}$):

$$\frac{\dot{k}}{k} = \left( \frac{\dot{k}}{k} \right)_{k=k_{SS}} \left. + \frac{d (\dot{k}/k)}{d \ln k} \right|_{k=k_{SS}} (\ln k - \ln k_{SS}) + O(2)$$

$$\approx 0 + k \frac{d (\dot{k}/k)}{dk} \left|_{k=k_{SS}} \right. \ln (k/k_{SS})$$

$$\approx k_{SS} (\alpha - 1) sAk_{SS}^{\alpha-2} \ln (k/k_{SS})$$

$$\approx (\alpha - 1) (n + \gamma + \delta) \ln (k/k_{SS})$$

$$= -\beta$$

The absence of $s$ in the expression above is a particular feature of the Cobb-Douglas case. On the one hand, given $k$, a higher savings rate leads to greater investment and thus speeds up convergence; on the other hand, it increases $k_{SS}$ and thus reduces the average product of capital close to the steady state, slowing down convergence. With the Cobb-Douglas technology, these two counteracting effects exactly cancel. The story for $A$ is analogous.

If you want to calculate the convergence rate for the general CES case, you will find help in chapter 1.3.4 of BSiM, the savings rate will influence the steady state convergence rate.
5. The reduction in \( \gamma \) shifts the \((n + \gamma + \delta)k\)-line downwards, leading to an intersection with the \(sf(k)\)-curve to the right of the old steady state (in terms of capital per efficiency units). Thus, \( k_{SS}^1 > k_{SS}^0 \). Hence, at \( t = T \), i.e. at the occurrence of the shock, the growth rate of effective capital per worker \( k \) jumps to \( \gamma_0 - \gamma_1 > 0 \) and declines gradually as \( k_t \) expands to reach 0 at the new steady state (see equation (6)).

What occurs at \( T \)? In the old steady state, \( sf(k)/k = (n + \gamma_0 + \delta) \). An instant later, the capital stock is still the same \( k_T \), so when using \( g_k = sf(k)/k - (n + \gamma_1 + \delta) \) we obtain at \( t = T \) \( g_k|_{t=T} = \gamma_0 - \gamma_1 > 0 \). \( g_k \) decreases over time as capital in efficiency units builds up and the decreasing marginal returns to capital generate less output.

The growth rate of the per-capita capital stock \( K/N \) is \( g_{K/N} = g_k + \gamma \). After the shock \( \gamma = \gamma_1 \). Combining with the result for \( g_k \) at \( t = T \) we obtain: \( g_{K/N}|_{t=T} = \gamma_0 - \gamma_1 + \gamma_1 = \gamma_0 \).

Hence, there is no discontinuity for the per-capita capital growth rate at \( t = T \). In the following it declines as \( g_k \) declines.

6. The five Kaldor facts presented in the lecture are explained by the simple Solow model.

(a) Per capita output is defined by \( Y/N \) or alternatively by \( X \frac{Y}{XN} = Xy \). The growth rate \( g_{Y/N} = g_X + g_y \). At the steady state (Balanced Growth Path), we have \( g_y = 0 \), due to \( g_k = 0 \), but technology \( X \) grows with the constant rate \( \gamma \). Hence \( g_{Y/N} = \gamma \) on the BGP.

(b) Per capita capital is defined by \( K/N \) or by \( Xk \). Its growth rate is \( g_{K/N} = g_X + g_k \) which is equal to \( \gamma \) at the balance growth path.

(c) The ratio of physical capital to output is \( K/Y \). For this to be constant we need \( g_{K/Y} = g_K - g_Y = 0 \): capital and output need to grow in equal terms. Again, \( K = XNk \) and \( Y = XNy \), so \( g_K = g_X + g_N + g_k \) and \( g_Y = g_X + g_N + g_y \). In steady state \( g_k = g_y = 0 \) and hence \( g_K = g_Y = n + \gamma \).
(d) The rate of return is defined by \( r + \delta = f'(k) \). As \( k_{SS} \) is constant, we have that \( r + \delta \) is constant. In the case of constant depreciation \( r \) must also be constant. In the Cobb-Douglas case \( r = Aa^{\frac{\alpha}{1-\alpha}} - \delta \).

(e) Share of labour is defined as \( wN/Y \) and the capital share is \( (r + \delta)K/Y \), which are equal to \( 1 - \alpha \) and \( \alpha \), respectively, in the Cobb-Douglas case. We have shown that in steady state, wages \( w \) grow at rate \( \gamma \) (see equation (18)), we know that population grows at rate \( n \) and output grows at the rate of technology and population \( g_Y = \gamma + n \). Labour share grows at rate \( g_{LSh} = g_w + g_N - g_Y = \gamma + n - (\gamma + n) = 0 \). Regarding the capital share, we have in steady state that the interest rate is constant and capital \( K \) and output both grow at rate \( \gamma + n \). Hence \( g_{KSh} = 0 \).

Kaldor’s facts are not all independent of each other. By employing the third and the fifth fact to eliminate the fourth one, we can see that, if the capital share is constant \( rK/Y \) and the capital-output ratio is constant \( K/Y \), then \( r \) needs to be constant as well.

7. Golden rule level of \( k \) is the value of capital in efficiency units for which the economy is at the BGP with maximized level of steady-state consumption. The Golden rule condition that needs to be satisfied is given by \( f'(k^G) = n + \delta + \gamma \), which in the Cobb-Douglas case becomes

\[
\alpha A k^{\alpha-1} = n + \delta + \gamma
\]

With income tax rate \( \tau \), the economy is saving \( s(1 - \tau) \) of its income every period, so the law of motion for capital becomes

\[
\dot{K} = s(1 - \tau)Y - \delta K, \text{ or }
\]

\[
\dot{k} = s(1 - \tau)y - (n + \delta + \gamma)k
\]

when transformed into units of effective labor. Dividing both sides by \( k \) and using the BGP condition \( (\dot{k} = 0) \), one obtains

\[
\frac{s(1 - \tau)}{\alpha} A k^{\alpha-1} = n + \delta + \gamma
\]

\[
\frac{s}{\alpha} (1 - \tau) \alpha A k^{\alpha-1} = n + \delta + \gamma.
\]

To achieve dynamic efficiency, the tax rate needs to be such that the Golden rule condition is satisfied, so

\[
\frac{s}{\alpha} (1 - \tau) = 1
\]

\[
\tau = \frac{s - \alpha}{s}
\]
To interpret this rate, notice that in the set-up without government, a saving rate equal to $\alpha$ puts the economy at the Golden rule BGP. Since $\alpha$ is the capital share of income, it implies that an economy which saves its total capital income and consumes total labor income is dynamically efficient:

$$\dot{K} = \alpha Y - \delta K.$$  \hspace{1cm} or \hspace{1cm} $$\dot{k} = \alpha y - (n + \delta + \gamma)k$$

and with the BGP condition ($k = 0$), one obtains

$$\alpha Ak^{\alpha-1} = n + \delta + \gamma.$$

So, $\alpha$ is the golden rule saving rate and the optimal tax derived above can now be interpreted as the share of the economy’s saving rate which exceeds the level $\alpha$. In other words, to reach the Golden rule path, the government needs to tax away (and spend) the amount of resources by which current savings exceed $\alpha$ share of total income.

2 CES Technology

(You can find helpful information on this exercise in chapter 1 and its appendix in Barro & Sala-i-Martin (BSiM). If you want to know more about the limit behavior of the CES production function, you can consult Varian for example.)

1. The production function in extensive form is

$$F(K, N) = [\alpha K^\rho + (1 - \alpha)N^\rho]^{1/\rho}$$ \hspace{1cm} (22)

where $0 < \alpha < 1$, $\rho < 1$, and $\rho \neq 0$. To write it in intensive form, define $k = K/N$, divide both sides of (22) by $N$, and rearrange:

$$\frac{F(K, N)}{N} = [\frac{\alpha K^\rho + (1 - \alpha)N^\rho}{N}]^{1/\rho}$$

$$F(\frac{K}{N}, 1) = \left[\frac{\alpha K^\rho + (1 - \alpha)N^\rho}{N^\rho}\right]^{1/\rho}$$

$$f(k) \equiv [\alpha k^\rho + (1 - \alpha)]^{1/\rho}$$ \hspace{1cm} (23)

To check assumptions A1 to A3, we need to have expressions for the first two derivatives
of (23). An expression for the average product \( f(k)/k \) will also be useful. These are

\[
\begin{align*}
\frac{f'(k)}{k} &= \alpha k^{\rho - 1} \left[ \alpha k^{\rho} + (1 - \alpha) \right]^{1-\rho}, \\
&= \alpha [k^{1-\rho} (\alpha k^{\rho} + 1 - \alpha)]^{1-\rho}, \\
&= \alpha [\alpha + (1 - \alpha) k^{\rho}]^{1-\rho}. \tag{24}
\end{align*}
\]

\[
\frac{f(k)}{k} = \frac{[\alpha k^{\rho} + (1 - \alpha)]^{1/\rho}}{k}
\]

\[
= \frac{[k^{1-\rho} (\alpha k^{\rho} + 1 - \alpha)]^{1/\rho}}{k}
\]

\[
= [\alpha + (1 - \alpha) k^{\rho}]^{1/\rho}. \tag{25}
\]

\[
\frac{f''(k)}{k} = -\alpha (1 - \alpha) (1 - \rho) k^{-\rho - 1} [\alpha + (1 - \alpha) k^{-\rho}]^{\frac{1-2\rho}{\rho}}. \tag{26}
\]

Now the assumptions can be checked.

**A1:** By inspection of (23), (24), and (26), and by the restrictions on the parameters, it is immediately clear that A1 holds for all \( k > 0 \). Marginal and average product are positive and diminishing in \( k \) for all values of \( \rho < 1 \).

**A2:** Here, two cases have to be distinguished. First consider a high degree of substitution between \( K \) and \( N \), i.e. \( 0 < \rho < 1 \). Then the limits of the marginal and average product of capital are

\[
\begin{align*}
\lim_{k \to 0^+} f'(k) &= \infty, \\
\lim_{k \to \infty} f'(k) &= \alpha^{1/\rho} > 0 \tag{27}
\end{align*}
\]

Observe that a significant difference to the Cobb-Douglas case arises: As \( k \) goes to infinity, its marginal and average product do not vanish, but approach a positive constant. As we see later, this makes endogenous growth in the steady state possible.

Now consider a low degree of substitution between \( K \) and \( N \), i.e. \( \rho < 0 \). Then the limits are

\[
\begin{align*}
\lim_{k \to 0^+} f'(k) &= \alpha^{1/\rho} < \infty, \\
\lim_{k \to \infty} f'(k) &= 0 \tag{28}
\end{align*}
\]

Again, not all of the the Inada conditions are fulfilled. In this case, it may arise that the marginal product near \( k = 0 \) is not sufficiently large to lead to a strictly positive steady state value of \( k \) (again see below for details).

**A3:** \( \lim_{k \to 0^+} f(k) = (1-\alpha)^{1/\rho} > 0 \) in the high substitution case \( (0 < \rho < 1) \), implying that not every factor is essential for production as in the Cobb-Douglas. \( \lim_{k \to 0^+} f(k) = 0 \) in the low substitution case \( \rho < 0 \).
The elasticity of substitution is defined as

\[ \sigma = -\frac{d \left( \frac{K}{N} \right)}{d \left( \frac{F_K}{F_N} \right)} \frac{d \left( \frac{K}{N} \right)}{d \left( \frac{F_K}{F_N} \right)} \]

where \( F_K \) and \( F_N \) represent the derivative of \( F(K,N) \) with respect to \( K \) and \( N \), respectively. It reflects the change of relative factor utilization when the relative productivities of the factors change. In that sense it reflects the sensitivity of the input mix to the changes in the relative prices of inputs (only with full competition). To show that \( \sigma \) is constant in the case of a CES production function, note that

\[ \sigma = -\frac{d \ln \left( \frac{K}{N} \right)}{d \ln \left( \frac{F_K}{F_N} \right)} = -\frac{d \ln \left( \frac{K}{N} \right)}{d \ln (\text{TRS})} \]

Let us define various elements:

\[ d \ln \left( \frac{K}{N} \right) = \frac{\partial \ln \left( \frac{K}{N} \right)}{\partial \left( \frac{K}{N} \right)} d \left( \frac{K}{N} \right) \]

\[ d \ln \left( \frac{F_K}{F_N} \right) = \frac{\partial \ln \left( \frac{F_K}{F_N} \right)}{\partial \left( \frac{F_K}{F_N} \right)} \frac{\partial \left( \frac{F_K}{F_N} \right)}{\partial \left( \frac{K}{N} \right)} d \left( \frac{K}{N} \right) \]

\[ \frac{F_K}{F_N} = \frac{\alpha}{1 - \alpha} \left( \frac{K}{N} \right)^{\rho-1} \]

With these definitions we obtain:

\[ \sigma = -\frac{d \ln \left( \frac{K}{N} \right)}{d \ln \left( \frac{F_K}{F_N} \right)} = -\frac{\frac{\partial \ln \left( \frac{K}{N} \right)}{\partial \left( \frac{K}{N} \right)} d \left( \frac{K}{N} \right)}{\frac{\partial \ln \left( \frac{F_K}{F_N} \right)}{\partial \left( \frac{F_K}{F_N} \right)} \frac{\partial \left( \frac{F_K}{F_N} \right)}{\partial \left( \frac{K}{N} \right)} d \left( \frac{K}{N} \right)} \]

\[ = -\frac{\frac{1 - \alpha}{\alpha} \left( \frac{K}{N} \right)^{1-\rho} \frac{\alpha}{1 - \alpha} \left( \frac{\rho}{K} \right)^{\rho-2} d \left( \frac{K}{N} \right)}{\frac{1}{\left( \frac{\rho}{K} \right)^{\rho-2} \left( \frac{K}{N} \right)^{\rho-1}}} = \frac{1}{1 - \rho} \]

2nd alternative, if you want to go the hard way: We will use equations (5), (24), (26), and
\( \ln x(y) = \frac{1}{x(y)} \frac{\partial x(y)}{\partial y} \) and \( k = K/N. \)

\[
\sigma = - \frac{d \ln (K/N)}{d \ln (F_K/F_N)} = - \frac{d \ln (F_k/(f-kf_k))}{d \ln (F_k/(f-kf_k))} = - \frac{f_k (f-kf_k)}{kf_{kk}f_k} \left[ \frac{1}{k} - \frac{f_k}{f} \right]
\]

\[
= - \frac{\alpha (\alpha + (1-\alpha)k^{-\rho})^{1-\rho}}{-\alpha(1-\alpha)(1-\rho)k^{-\rho-1}[\alpha + (1-\alpha)k^{-\rho}]^{1-\rho}} \cdot \left[ \frac{1}{k} - \frac{\alpha k^{\rho-1}[(\alpha k^{\rho} + (1-\alpha)]^{1-\rho}}{[\alpha k^{\rho} + (1-\alpha)]^{1-\rho}} \right]
\]

\[
= \frac{\alpha k^{\rho} + (1-\alpha)}{(1-\alpha)(1-\rho)} \left[ \frac{1}{k} - \frac{\alpha k^{\rho-1}}{\alpha k^{\rho} + (1-\alpha)} \right]
\]

\[
= \frac{\alpha k^{\rho} + (1-\alpha) - \alpha k^{\rho}}{(1-\alpha)(1-\rho)} = 1/(1-\rho)
\]

If the parameter \( \rho \) is constant, then the elasticity of substitution of a CES production function is independent of the level \( k \), hence \( \sigma = 1/(1-\rho) \) is constant.

3rd alternative solution (see BSiM, 1.5.4).

2.4. The growth rate of \( k \) is given by (we abstract from technological progress)

\[
\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (\delta + n). \tag{31}
\]

To analyze this expression, first consider the limit behavior of the average product \( f(k)/k \), distinguishing again between two cases for the parameter \( \rho \). (In the following, it is useful to work with a graph plotting the two parts of (31) against \( k \).)

\( 0 < \rho < 1: \)

\[
\lim_{k \to 0^+} f(k)/k = \infty \quad \lim_{k \to \infty} f(k)/k = \alpha^{1/\rho} > 0. \tag{32}
\]

Now two cases are possible. When \( s \alpha^{1/\rho} > \delta + n \) (as in question 2.3), the growth rate of capital per capita (31) is always positive. There exists a balanced growth path with endogenous growth of the per capita capital stock at the rate

\[
(g_k)_{ss} = \left( \frac{k}{k} \right)_{ss} = s \alpha^{1/\rho} - (\delta + n). \tag{33}
\]

The violation of the Inada condition for \( k \to \infty \) already indicated this: the marginal product of capital is diminishing, but does not vanish. Therefore, additions to the capital stock
remain productive, and endogenous growth purely by capital accumulation is possible. (see also the discussion in chapter 1.3.3 (BSiM).)

An alternative solution, knowing that the growth rate is constant, would be to set the time derivative of capital growth to zero:

\[ \frac{\partial g}{\partial t} = 0; \]

\[ \dot{g}_k = \frac{\partial g}{\partial t} = \frac{\partial (\dot{k}/k)}{\partial t} = s \frac{k}{k} \left[ f'(k) - \frac{f(k)}{k} \right] \]  

Due to the fact that \( \dot{k}/k \neq 0 \forall t \), a necessary condition for constant growth is \( f'(k) - \frac{f(k)}{k} = 0 \), which is indeed fulfilled for \( t \to \infty \) because \( k \to \infty \) and \( \lim_{k \to \infty} f'(k)/k = \alpha^{1/\rho} \).

In order to find a variable that exhibits zero growth rate in steady state, try \( z = f(k)/k \):

\[ g_z = \dot{z}/z = \dot{k}/k \left[ f'(k) - \frac{f(k)}{k} \right] \]

\[ \lim_{k \to \infty} g_z = 0, \text{ because } \lim_{k \to \infty} f'(k) = \lim_{k \to \infty} \frac{f(k)}{k} = \alpha^{1/\rho} \]

When however \( s\alpha^{1/\rho} < \delta + n \), there is a unique steady state with a constant stock of capital per capita. Setting the left-hand side of (31) to zero, the steady-state capital stock turns out to be

\[ k_{SS} = \left[ \frac{s^\rho(1-\alpha)}{(n+\delta)^\rho - s^\rho \alpha} \right]^{1/\rho} = \left[ \frac{(\delta + n)^\rho}{(1-\alpha)s^\rho - \frac{\alpha}{1-\alpha}} \right]^{1/\rho}. \]  

(Because \( s\alpha^{1/\rho} < \delta + n \), this is positive.) In this case, although the marginal product of capital does not go to zero as \( k \to \infty \), it goes to a value that is smaller than that needed
just to maintain capital per capita constant when depreciation and population growth rates are positive. So no endogenous growth in this case.

\( \rho < 0 \):

Consider the limit behavior of the average product to draw conclusions about the growth rate using (31). Now,

\[
\lim_{k \to 0^+} \frac{f(k)}{k} = \alpha^{1/\rho} < \infty \tag{36}
\]

\[
\lim_{k \to \infty} \frac{f(k)}{k} = 0. \tag{37}
\]

We already know from the verification of A2 that in this case, there can be no endogenous growth since the marginal product of capital vanishes as \( k \to \infty \). This is borne out by (36). So when \( s\alpha^{1/\rho} > \delta + n \), the economy again converges to a steady state with zero growth and the capital stock per capita given by (35). (In this case the conditions \( s\alpha^{1/\rho} > \delta + n \) and \( \rho < 0 \) ensure that \( k_{SS} > 0 \).)

A more peculiar case arises when \( s\alpha^{1/\rho} < \delta + n \) (as in question 2d)). In this case (31) shows that the growth rate of \( k \) is negative for all \( k \). Then there is no steady state with positive capital. Even when capital is zero, the combination of its marginal product and the savings rate is not enough to compensate for depreciation and population growth. So whatever \( k_0 \), capital keeps falling until the economy reaches a steady state with \( k = 0 \).

In all cases with a steady state with positive \( k \), convergence properties can easily be deduced from (31). Since \( f(k)/k \) decreases in \( k \), this is also the case for \( \dot{k}/k \). So an economy that is converging from \( k_0 < k_{SS} \) to \( k_{SS} \) exhibits a decreasing growth rate the closer it gets to the steady state. In the endogenous growth case, the growth rate is equally decreasing in \( k \), but approaches the positive constant given in (33). To summarise

<table>
<thead>
<tr>
<th>( \text{cond. on } \rho )</th>
<th>( \text{cond. on } s )</th>
<th>( \lim_{t \to \infty} g_y(k) ) or ( k_{SS} ) if it exists</th>
<th>marginal product of capital does not tend to zero, it is sufficiently high to sustain end. growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; \rho &lt; 1 )</td>
<td>( s\alpha^{1/\rho} &gt; \delta + n )</td>
<td>( s\alpha^{1/\rho} - (\delta + n) )</td>
<td>marginal product of capital does not tend to zero, it is sufficiently high to sustain end. growth</td>
</tr>
<tr>
<td>( s\alpha^{1/\rho} &lt; \delta + n )</td>
<td>( \frac{(\delta+n)^\rho}{(1-\alpha)^\rho} - \frac{\alpha}{1-\alpha} )</td>
<td>( -1/\rho )</td>
<td>Savings too low to sustain end. growth although marginal product of capital does not tend to zero</td>
</tr>
<tr>
<td>( \rho &lt; 0 )</td>
<td>( s\alpha^{1/\rho} &gt; \delta + n )</td>
<td>( \frac{(\delta+n)^\rho}{(1-\alpha)^\rho} - \frac{\alpha}{1-\alpha} )</td>
<td>Factors are too complementary to generate endogenous growth. Marg. prod at ( k=0 ) large enough to allow positive steady state</td>
</tr>
<tr>
<td>( s\alpha^{1/\rho} &lt; \delta + n )</td>
<td>0</td>
<td>0</td>
<td>Factors are too complementary to generate endogenous growth. Savings too low, depreciation larger than investment even ( k = 0 )</td>
</tr>
</tbody>
</table>

5.-6. The elasticity of substitution between capital and labour is \( 1/(1 - \rho) \). Hence \( 0 < \rho < 1 \) implies high values of the elasticity (capital and labour are substitutes), \( \rho < 0 \) low ones (they are complements). It is intuitive that a low elasticity of substitution tends to constrain production possibilities. The problem is that as capital is accumulated, but labour remains constant, the production technology cannot sufficiently substitute capital
for labour. Output is constrained through the fact that labour does not accumulate. In the limit case $\rho \to -\infty$, the production function approaches a Leontief (fixed-proportions) production function. Marginal products for each factor individually are zero for all $K$ and $N$, so increasing capital per worker does not raise output. Instead, if substitution is high, labour is substituted by capital as the latter accumulates. Consider the limit cases: For $\rho = 1$, the production function is linear, and capital and labour are perfect substitutes. Marginal products are constant, and the model corresponds to the $AK$ model, a model of endogenous growth. Remark: As $\rho \to 0$, the production function approaches the Cobb-Douglas case (derivation in 1.5.4 of BSiM, take logs of $Y$ to get $\lim_{\rho \to 0} (\log (Y)) = \lim_{\rho \to 0} \left( \frac{\log [\alpha K^\rho + (1-\alpha)N^\rho]}{\rho} \right)$ and then use l’Hôpital’s rule).