# Microeconomics, Block I
# Part 2

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Oct. 2, 2017

# Pure Exchange Economy

- $H$ consumers with:
  - preferences described by $U^h : \mathbb{R}^L_+ \to \mathbb{R}$ and
  - resources $\omega^h \in \mathbb{R}^L_+$, $h = 1, .., H$ (income comes from the sale of individual endowment of resources).

- *Economic Problem/Activity*: (re-)allocate resources among the consumers

  - **allocation:** array $\mathbf{x} = (x^1, .., x^H) \in \left(\mathbb{R}^L_+\right)^H$
    - **feasible** if it is consistent with available resources: $\sum_h x^h \leq \sum_h \omega^h$

      In the special case $L = 2$, $H = 2$, feasible allocations can be graphically represented using the *Edgeworth Box.*

# Pareto Efficiency

- Evaluate alternative allocations using agents' preferences:

Pareto Efficiency (PE): An allocation $\mathbf{x} \in \left(\mathbb{R}^L_+\right)^H$ is Pareto efficient if:
(i) it is feasible
(ii) there is no other feasible allocation $\hat{\mathbf{x}} \in \left(\mathbb{R}^L_+\right)^H$ which allows to improve agents' welfare (weakly for all, strictly for at least one):

$$\begin{aligned} U^h(\hat{x}^h) &\geq U^h(x^h) \text{ for all } h, \\ U^{h'}(\hat{x}^{h'}) &> U^{h'}(x^{h'}) \text{ for at least some } h' \end{aligned}$$

- When the above inequalities hold, we say $\hat{\mathbf{x}}$ *Pareto dominates* $\mathbf{x}$.

# Pareto Efficiency cts.

Note: Pareto dominance defines a strict (social) preference relation over the set of allocations $\left(\mathbb{R}^L_+\right)^H$.
A weak social preference relation is analogously defined; it is however incomplete *[why?]*.

$$\Downarrow$$

- Hence, a solution of the social (multi-agent) choice problem of finding a feasible allocation that weakly Pareto dominates all other feasible allocations cannot generally be found.
- On the other hand, there are typically (many) feasible allocations that are not Pareto dominated by some other feasible allocation: Pareto efficiency is still a useful criterion to evaluate allocations.

# How to find PE allocations

- PE allocations are solutions of the problem:

$$\max_{\mathbf{x}} U^h(x^h) \\ \text{s.t.} \left\{ \begin{array}{c} U^{h'}(x^{h'}) \geq \bar{U}^{h'} \text{ for all } h' \neq h \\ \sum_h x^h \leq \sum_h \omega^h \end{array} \right. \tag{1}$$

for some $h$ and some given $\left(\bar{U}^{h'}\right)_{h' \neq h}$.

FOCs (for an interior solution, under $A.1'$ and differentiability of $U^h$ for all $h$):

$$\begin{aligned} DU^h &= \rho \\ \mu^{h'} DU^{h'} &= \rho \text{ for all } h' \neq h \\ \sum_h x^h &= \sum_h \omega^h \end{aligned}$$

where $\left(\mu^{h'}\right)_{h' \neq h}$ and $\rho$ are the Lagrange multipliers of the two sets of constraints in (1).
Under $A.2$ FOCs are also sufficient conditions for a Pareto optimum.

Thus:

$$DU^h = \mu^{h'} DU^{h'} \text{ for all } h' \neq h$$

utility gradients are collinear for all agents (or: MRS are equalized across agents)

- Varying the values of $\bar{U}^{h'}$ for $h' \neq h$ the solution typically changes $\Rightarrow$ Obtain set of all Pareto efficient allocations.
- Finding PE allocations in the Edgeworth box.
  Examples:

..

# Utility Possibility set and Welfare weights

- **Utility possibility set:** image of feasible allocations in the space of utility levels

$$UP = \left\{ \left(U^h\right)_{h=1}^{H} = \left(U^h(x^h)\right)_{h=1}^{H};\ \text{for } \mathbf{x} \in \left(\mathbb{R}_+^L\right)^H,\ \sum_h x^h \leq \sum_h \omega^h \right\}$$

PE allocations support points on the outer (NorthEast) boundary of UP:

$$\begin{aligned} \left(U^h\right)_{h=1}^{H} &\in UP : \\ \nexists \left(\hat{U}^h\right)_{h=1}^{H} &\in UP \text{ s.t. } \hat{U}^h \geq U^h,\ > \text{ for some } h \end{aligned}$$

Under what conditions can we say UP is closed? and convex? and has a downward sloping outer boundary? [see HW]

# Utility Possibility set and Welfare weights cts.

- PE allocations can also be found (under a mild extra condition) as solutions of the following problem of maximizing social welfare:

$$\max_{\mathbf{x}} \sum_h \xi^h U^h(x^h)$$
$$\text{s.t. } \sum_h x^h \leq \sum_h \omega^h$$

for any given set of strictly positive (welfare) weights $\left(\xi^h\right)_{h=1}^{H}$.

equivalent to finding a point on the outer boundary of UP

# Competitive Markets

- Various mechanisms are possible to attain a reallocation of existing resources:
  - bargaining,
  - voting,
  - dictatorship,
  - markets, ...
- Will focus on the latter: allocations attained when agents trade in perfectly competitive markets.
  - $p$ such that $\mathbf{x} = \left(.., x^h(p, p \cdot \omega^h), ..\right)$ is a feasible allocation (note: $m^h = p \cdot \omega^h$).

# Competitive Equilibrium

More formally:

Competitive Equilibrium: is a price vector $p$ and an allocation $\mathbf{x} = (.., x^h, ..)$ such that:
(i) for all $h$, $x^h = x^h(p, p \cdot \omega^h)$, and
(ii) $\mathbf{x}$ is feasible.

That is: consumers optimize and markets clear.

- Finding CE allocations in the Edgeworth box:
    - draw the *offer curves* (consumption bundles demanded at some price) for each consumer.
      *[properties?]*
    - CE obtains when offer curves intersect (except at endowment point)

# Competitive Equilibrium in Production Economies

- representative firm with technology $Y \subset \mathbb{R}^L$
- *Allocation*: $(y, \mathbf{x}) \in Y \times \left(\mathbb{R}^L_+\right)^H$
  - *feasible* if: $\sum_h x^h \leq \sum_h \omega^h + y$

  - *Pareto efficient*: def. unchanged (need only to replace feasibility notion on p.2 with the one above)

FOCs for Pareto efficiency: must add the following condition (when $Y$ is representable by a differentiable production function, with commodity 1 as output and all other goods as inputs: $y_1 = f(z_2, .., z_L)$):

$$\rho_1 \frac{\partial f}{\partial z_j} = \rho_j \text{ for } j = 2, ..L$$

Competitive Equilibrium with Production: is a price vector $p$ and an allocation $\bar{y}, \mathbf{x}$ such that:
(i) for all $h$, $x^h = x^h(p, p \cdot \omega^h + \theta^h \pi)$,
(ii) $\bar{y} \in \arg\max_{y \in Y} p \cdot y$, and $\pi = p \cdot \bar{y}$
(iii) $\bar{y}, \mathbf{x}$ is feasible.

# Welfare Properties: FWT

Trade is voluntary, hence equilibrium allocations will satisfy individual rationality:

$$U^h(x^h) \geq U^h(\omega^h) \text{ for all } h, \qquad \text{and more:}$$

- First Welfare Theorem: Under the Assumptions ..., all competitive equilibrium allocations are Pareto efficient

**Proof.**

Suppose $\mathbf{x} = (.., x^h, ..)$ is a competitive equilibrium allocation for some price $p$ and is not Pareto efficient. Then there must be another allocation $\hat{\mathbf{x}} = (.., \hat{x}^h, ..)$ which is feasible and Pareto improves upon $\mathbf{x}$. But since $x^h$ is the optimal choice of consumer $h$ at prices $p$, under ...
$U^h(\hat{x}^h) > U^h(x^h)$ implies $p \cdot \hat{x}^h > p \cdot \omega^h$ for all $h$ and also, under ...,
$U^h(\hat{x}^h) = U^h(x^h)$ implies $p \cdot \hat{x}^h \geq p \cdot \omega^h$, with all the previous inequalities being strict for at least some $h$. Summing the latter inequality over $h$ yields $p \cdot \sum_h \hat{x}^h > p \cdot \sum_h \omega^h$ which contradicts the feasibility of $\hat{\mathbf{x}}$ (since, under ..., $p \geq 0$). □

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  - economy where markets for some of the $L$ commodities are not open.
  - economy with externalities:

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  - economy where markets for some of the $L$ commodities are not open.
  - economy with externalities:
    - completeness - and hence efficiency - can be restored in this case by suitably expanding set of markets: *Lindahl Equilibria*.

# Gains from Trade

Consider two countries:
country $A$ with $H^A$ consumers and country $B$ with $H^B$ consumers (and the same $L$ goods in each of them)

- *Autarky equilibrium*: $p^A, \mathbf{x}^{Aaut}$ such that country A markets (where only consumers of that country can trade) clear and $p^B, \mathbf{x}^{Baut}$ such that country B markets (where only country B consumers can trade) clear:

$$\sum_{h^A=1}^{H^A} x^{h^A} \leq \sum_{h^A=1}^{H^A} \omega^{h^A}; \ \sum_{h^B=1}^{H^B} x^{h^B} \leq \sum_{h^B=1}^{H^B} \omega^{h^B}$$

- *Free trade equilibrium*: $p, \mathbf{x}^A, \mathbf{x}^B$ such that international markets (where consumers of both countries trade) clear:

$$\sum_{h^A=1}^{H^A} x^{h^A} + \sum_{h^B=1}^{H^B} x^{h^B} \leq \sum_{h^A=1}^{H^A} \omega^{h^A} + \sum_{h^B=1}^{H^B} \omega^{h^B}$$

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    - no

**Proof.**

Suppose not: $U^{h^A}(x^{h^{Aaut}}) \geq U^{h^A}(x^{h^A})$ for all $h^A = 1, .., H^A$ (with a strict inequality for some $h^A$). Then $p \cdot x^{h^{Aaut}} \geq p \cdot x^{h^A}$, with a strict inequality for some $h^A$. Since $p \cdot x^{h^A} = p \cdot \omega^{h^A}$ for all $h^A$ (again assuming A.1), summing over $h^A$ yields $p \cdot \left(\sum_{h^A} x^{h^{Aaut}}\right) > p \cdot \left(\sum_{h^A} \omega^{h^A}\right)$. But this contradicts feasibility of $\mathbf{x}^A$: $\sum_{h^A} x^{h^{Aaut}} \leq \sum_{h^A} \omega^{h^A}$. □

- Still can say little on distribution of gains and losses.
  Can we find a system of lump sum net transfers $\left(t^{h^A}\right)_{h^A}, \left(t^{h^B}\right)_{h^B}$ to the agents in each country such that: (i) budget balance holds in each country ($\sum_{h^A} t^{h^A} = 0$, $\sum_{h^A} t^{h^B} = 0$), and (ii) all agents (weakly) gain when going from autarky to free trade?

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  - yes (Grandmont - Mc Fadden 1972)

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**Proof.**

[sketch] Set $t^{h^A} = x^{h^{Aaut}} - \omega^{h^A}$, $t^{h^B} = x^{h^{Baut}} - \omega^{h^B}$ for all $h^A, h^B$. Budget balance then clearly holds (since $\mathbf{x}^A, \mathbf{x}^B$ satisfy market clearing in each country). The Pareto improvement property then follows from the individual rationality of the competitive equilibrium (with free trade, relative to the after transfers endowment, given by $x^{h^{Aaut}}$ for any $h^A$ and by $x^{h^{Baut}}$ for any $h^B$). □

# SWT

- Second Welfare Theorem: Assume individual preference are strictly monotone and convex. Then every Pareto efficient allocation **x** can be decentralized as a competitive equilibrium with lump sum transfers.

That is, for any Pareto efficient **x**, we can find some lump sum transfers $t^h \in \mathbb{R}^L$, $h = 1, ..H$, such that $\sum_h t^h = 0$ and the economy where consumers have initial endowment $\omega^h + t^h$ has a competitive equilibrium given by $p, \mathbf{x}$.

**Proof.**

Set $t^h = x^h - \omega^h$ for all $h$. Consider the following two sets: the singleton $\{\omega\}$, where $\omega \equiv \sum_h \omega^h = \sum_h x^h$ and $V \equiv \sum_h SP^h(x^h)$ (the sum of the strictly preferred sets of each individual, evaluated at their consumption at the PE allocation). The second set is also convex (by the convexity of individual preferences).

Hence, by the Separating Hyperplane Theorem $\exists p \in \mathbb{R}^L$, $p \neq 0$ and $c \in \mathbb{R}$: $p \cdot z \geq c \; \forall z \in V$ and $p \cdot \omega \leq c$.

By strict monotonicity, $\omega \in cl\{V\}$, hence $p \cdot \omega = c$.

Also, for any $\hat{x}^h \succeq^h x^h$ we have (again by strict monotonicity of preferences) $p \cdot \hat{x}^h \geq p \cdot x^h$ for all $h$ [why?]. Thus $p, (x^h)_h$ is a *quasi-equilibrium* (that is, at $p$, $x^h$ solves the expenditure minimization problem for all $h$ and markets clear). □

**Proof.**

(cts.) If $p \cdot x^h > 0$, then $p \cdot \hat{x}^h > p \cdot x^h$, that is $p, (x^h)_h$ is also an equilibrium.[why?]
The strict monotonicity of preferences then implies that $p \gg 0$ so that $x^h > 0 \Rightarrow p \cdot x^h > 0$. □

# Existence of Equilibria

- Competitive equilibrium prices are solutions of the following system of equations (under strict monotonicity):

$$z\left(p; \left(\omega^h\right)_h\right) \equiv \sum_h x^h(p, p \cdot \omega^h) - \sum_h \omega^h = 0$$

$L$ equations in $L$ unknowns ($p$).

  - By Walras law (assuming $A.1$),

$$p \cdot \left( \sum_h x^h(p, p \cdot \omega^h) - \sum_h \omega^h \right) = 0 \text{ for all } p$$

    at most $L - 1$ equations are independent (can always omit market clearing equation for one market).
  - By homogeneity of degree zero in $p$ of individual, and hence aggregate demand, prices can always be normalized, e.g.

$$p \in \Delta^{L-1} \equiv \left\{ p \in \mathbb{R}^L_+ : \sum_l p_l = 1 \right\}: \text{ L-simplex (compact ad convex)}$$

- Equilibrium equations can thus be always reduced to $L-1$ equations in $L-1$ unknowns.
  Since equations are typically nonlinear, having number of unknowns less or equal than number of independent equations does not ensure a solution exists

To illustrate this, consider case $L=2$: suffices to consider the function $z_1(p_1)$
- Illustrate graphically that if $z_1(p_1)$ is not continuous, existence may fail

Note: what matters is continuity of aggregate demand. Can this be continuous when individual demand is not continuous? Yes, when we have economies with large number (infinitely many) consumers.

- Moreover, some 'boundary behavior' ($z_1(p_1) < 0$ for $p_1 \sim 1$ and $z_1(p_1) > 0$ for $p_1 \sim 0$) also needed for existence

- To show existence will use following

Fixed Point Theorem (Brouwer): If the map $f : S \to S$ is continuous, and the set $S$ is convex and compact, there exists a fixed point $x^*$: $f(x^*) = x^*$.

Simple illustration of result when $S = [0, 1]$:

As an immediate application of this result can show:

Theorem Let $z : \Delta^{L-1} \to \mathbb{R}^L$ be a continuous function, such that $p \cdot z(p) = 0$ for all $p$. Then there exists $p^*$ such that $z(p^*) \leq 0$.

**Proof.**

$$\text{Let } \varphi_l(p) = \frac{p_l + \max\{0, z_l(p)\}}{\sum_{j=1}^{L}\left[p_j + \max\{0, z_j(p)\}\right]},\ l = 1, ..L$$

Note that $\Delta^{L-1}$ is convex and compact, and $\varphi : \Delta^{L-1} \to \Delta^{L-1}$. Hence by above FPT, there is a fixed point $p^*$:

$$p_l^* = \frac{p_l^* + \max\{0, z_l(p^*)\}}{\sum_{j=1}^{L}\left[p_j^* + \max\{0, z_j(p^*)\}\right]},\ l = 1, ..L. \text{ Thus}$$

$$z_l(p^*)p_l^* \sum_{j=1}^{L}\left[p_j^* + \max\{0, z_j(p^*)\}\right] = z_l(p^*)p_l^* + z_l(p^*)\max\{0, z_l(p^*)\}$$

Summing over $l$ yields:

$$0 = \sum_l z_l(p^*)\max\{0, z_l(p^*)\} \Rightarrow z_l(p^*) \leq 0 \text{ for all } l.$$

- $p^*$ such that $z(p^*) \leq 0$ and Walras law ($p^* \cdot z(p^*) = 0$) imply that $z_l(p^*) < 0$ requires $p^*_l = 0$, but this is impossible under strict monotonicity. Hence:

$$z(p^*) = 0$$

- To properly claim existence of a competitive equilibrium need to face one last problem:
  under strict monotonicity, consumers' demand is not defined for prices on the boundary of $\Delta^{L-1}$ (when the price of some good is zero).
  $\Rightarrow$ use a limit argument (consider $z : \Delta^{L-1}_{\varepsilon} \rightarrow \mathbb{R}^L$, for $\varepsilon > 0$: $\Delta^{L-1}_{\varepsilon} \equiv \{p \in \mathbb{R}^L_+ : \sum_l p_l = 1,\ p_l \geq \varepsilon$ for all $l\}$, and take limit as $\varepsilon \rightarrow 0$)

# Other Existence Arguments

- With strictly convex preferences, a PE allocations maximizes, as we saw (over the set of feasible allocations), the linear social welfare function $\max_{\mathbf{x}} \sum_h \xi^h U^h(x^h)$ for some welfare weights $(\xi^h)_{h=1}^H$.
- Under the assumptions of the SWT, any PE allocation satisfies all the conditions for a competitive equilibrium except, possibly, the budget equation.

- Let $p(\xi)$ be the price vector supporting the PE allocation (shadow price of resource constraints) identified by the vector of weights $\xi$. $p(\xi)$ is also a competitive equilibrium price vector if:

$$p(\xi)(x^h(\xi) - \omega^h) = 0 \text{ for all } h = 1, .., H$$

This is a system of $H - 1$ independent equations in $H - 1$ unknowns (normalized weights $\xi$), convenient when $L$ is large relative to $H$ (e.g. in dynamic economies).

# How many Equilibria can there be?

Back to the case $L = 2$: simple graphical argument illustrates that:

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1. there can be many competitive equilibria, also a continuum of them;
2. situations with a continuum of equilibria are non robust to 'small perturbations' of the aggregate excess demand function;
3. uniqueness holds if demand is downward sloping.

# Local Uniqueness and Comparative Statics

- Can indeed show (using differential topology arguments, when utility functions are such that demand functions are continuous and differentiable) that, 'for almost all' economies (identified by endowment distribution $(\omega^h)_{h=1}^H$), competitive equilibria are locally isolated and - the $L-1 \times L-1$ matrix $|D_{\hat{p}}\hat{z}| \neq 0$, where $\hat{p} = (p_1, .., p_{L-1})$, $\hat{z} = (z_1, .., z_{L-1})$, and $p_L = 1$.

- Hence we can do **comparative statics analysis**: study how competitive equilibrium prices and allocations change in response to a change in the parameters, e.g. in the 'fundamentals of the economy', given by consumers' preferences and endowments, or in 'policy parameters' (taxes, ..).

# Comparative Statics

- Consider an infinitesimal change of the fundamentals, e.g. of the endowment of type 1 agents: $d\omega^1$.
  The effect on equilibrium prices can be obtained by applying the IFT to the system of $(L-1)$ equilibrium equations (in $L-1$ unknowns):
  $$\sum_h \left[\hat{x}^h(p, p \cdot \omega^h) - \omega^h\right] = 0$$
  if the matrix
  $$D_{\hat{p}}\hat{z} = \sum_h \left[D_{\hat{p}}\hat{x}^h + D_m\hat{x}^h \left(\hat{\omega}^h\right)^T\right]$$
  is invertible.
  In that case we have
  $$D_{\hat{\omega}^1}\hat{p} = \left(D_{\hat{p}}\hat{z}\right)^{-1}\left[d\hat{\omega}^1 - D_m\hat{x}^1\hat{p}^T d\hat{\omega}^1\right]$$
- Hence comparative statics effects crucially depend on $D_{\hat{p}}\hat{z}$, but we saw this matrix can be essentially arbitrary when $H$ is sufficiently large!

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- when aggregation holds (a representative consumer exists) for all $p$ and for given $(m^h)_{h=1}^H$
- when an appropriate generalization of the downward sloping demand condition to the case $L > 2$ (LOD) holds.

# Law of Demand and Uniqueness

1. If aggregate demand satisfies WARP - weaker than LOD - (and equilibria are locally isolated)

   [$z(p) = 0$ and $z(p') = 0$ imply, by WARP, that $z(\alpha p + (1-\alpha)p') = 0$ for all $\alpha$.
   This follows from the fact that Walras law implies that either $p \cdot z(\alpha p + (1-\alpha)p') \leq 0$ or $p' \cdot z(\alpha p + (1-\alpha)p') \leq 0$. But also $(\alpha p + (1-\alpha)p') \cdot z(p) \leq 0$, $(\alpha p + (1-\alpha)p') \cdot z(p') \leq 0$, hence by WARP $z(\alpha p + (1-\alpha)p') = z(p)$ for all $\alpha$.
   But this contradict the fact that equilibria are locally isolated.]

2. If aggregate demand satisfies the **gross substitute (GS) property:**

$$\text{for any } p, p' \text{ such that } p'_l > p_l \text{ and } p'_j = p_j \text{ for } j \neq l,$$
$$\text{we have: } z_j(p') > z_j(p) \text{ for all } j \neq l$$

(that is, an increase in the price of one good increases the demand in all other goods),
then there is at most one competitive equilibrium.

[Suppose $z(p) = z(p') = 0$. By homogeneity $z(\alpha p) = 0$ for all $\alpha > 0$. Set $\bar{\alpha} = \max_l p'_l / p_l$, so that $\bar{\alpha} p \geq p'$, applying GS yields a contradiction]

Note that GS implies that LOD holds at any equilibrium price
[GS holds for instance for Cobb Douglas and CES utility functions with $\sigma < 1$]

Consider market games:

- cooperative: core, ...
- Here will consider a **noncooperative market game** (Shapley-Shubik):
  - each consumer $h$ chooses supply $s_l^h \geq 0$ and bid $b_l^h \geq 0$ for each commodity $l = 1, .., L-1$, such that: $s_l^h \leq \omega_l^h$ for all $l < L$, $\sum_{l=1}^{L-1} b_l^h \leq \omega_L^h$
  - given the bids and supplies of all consumers, prices (in terms of commodity $L$) are determined as follows:

$$p_l(\mathbf{b}, \mathbf{s}) = \frac{\sum_h b_l^h}{\sum_h s_l^h}$$

and hence consumption levels:

$$x_l^h = \omega_l^h - s_l^h + \frac{b_l^h}{p_l(\mathbf{b}, \mathbf{s})}, \text{ for } l = 1, .., L-1$$

$$x_l^h = \omega_l^h - \sum_{l=1}^{L-1} b_l^h + \sum_{l=1}^{L-1} s_l^h p_l(\mathbf{b}, \mathbf{s})$$

# Market games

- Equilibrium: *Nash equilibrium* of the market game described

- need to specify level of prices in 'closed' markets, where $\sum_h b_l^h$ and/or $\sum_h s_l^h$ are zero.

- prices move 'against' trade of an agent: the more he bids for a good (say $l$), the lower will be $1/p_l(\mathbf{b}, \mathbf{s})$
- Budget set for agent $h$ is convex (when all markets are not 'closed').
  - as $H \to \infty$, if both $b_l^h / \sum_h b_l^h$ and $s_l^h / \sum_h s_l^h \to 0$ the budget set becomes linear in the limit and the Nash equilibrium coincides with the competitive equilibrium.
    Price taking behavior justified when the market share of each agent is 'negligible'

# Economy under Uncertainty

- individual **endowment**: $\widetilde{\omega}^h$, **random** variable, with (finite) support $\left(\omega^h(1), .., \omega^h(S)\right) \in \mathbb{R}^S_+$, for all $h$
  [consider, for simplicity, case where $L = 1$]

- State space $S = \{1, .., S\}$, $\pi = (\pi(1), .., \pi(S))$

- *allocation*: $\widetilde{x}^h$, also a *random* variable, with support $x^h = \left(x^h(1), .., x^h(S)\right) \in \mathbb{R}^S_+$ (defined on the same state space), for all $h$

- Preferences, now defined over random consumption bundles:

$$U^h : \widetilde{x} \to \mathbb{R}$$

  e.g. VNM preferences (expected utility): $\sum_s \pi(s) u^h(x(s))$.

# Allocation of Risk

- *Feasible allocation*: an allocation $\mathbf{x} = (.., \widetilde{x}^h, ..)$ such that:

$$\sum_h x^h(s) \leq \sum_h \omega^h(s) \text{ for all } s = 1, .., S$$

- Uncertainty only affects the fundamentals of the economy via the agents' endowments (no preference shocks)

- *Economic Problem*: allocation of (income) risk among the consumers Preferences describe individual attitude towards risk:
  - with VNM, strict convexity of preferences ($u^h : \mathbb{R}_+ \to \mathbb{R}$ strictly concave) implies agent is risk averse, i.e.

$$\mathbb{E}\widetilde{x} \succ^h \widetilde{x}$$

# Efficient allocations

- An allocation $\mathbf{x} = (.., \widetilde{x}^h, ..)$ is *Pareto efficient* if:
  - if it is feasible
  - there is no other feasible allocation that Pareto dominates it

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        - no, if agents have different beliefs $\pi^1 \neq \pi^2$ (→some betting/speculative trade is efficient in this case)
    - aggregate risk: $\sum_h \omega^h(1) \neq \sum_h \omega^h(2)$: full risk sharing not feasible, must also allocate aggregate risk.

# Markets

Individual preferences (under VNM): $\sum_s \pi(s) u^h(x(s))$

$$\Downarrow$$

- **Complete markets**: there exists one market for each $s = 1, ..S$ where agents can trade claims for delivery of the commodity contingent on the realization of that state

$$p = (p(1), .., p(S))$$

- *Timing* of markets also important: markets open before realization of the uncertainty

- Consumers' choice problem:

$$\max_x \sum_s \pi(s) u^h(x(s))$$
$$\text{s.t. } p \cdot x \leq p \cdot \omega$$

formally analogous to the one in deterministic case

# Competitive equilibria

- A **competitive equilibrium** is given by a price vector $p$ and an allocation $\mathbf{x}$ if the allocation solves the choice problem of every consumer $h$ at the prices $p$ and the allocation is feasible.

- Also formally the same as in deterministic case. Thus same properties hold: existence, FWT, SWT, ...

- *Examples* again:
  - with no aggregate risk and identical beliefs equilibrium is given by:

$$p(s) = \pi(s) \text{ for all } s,\ x^h(s) = \bar{x}^h = \sum_s \pi(s)\omega^h(s) \text{ for all } h$$

  - with no aggregate risk and $\pi^1(1) > \pi^2(1)$: $\pi^1(1) > p(1) > \pi^2(1)$ and $x^1(1) > x^1(2)$
  - with aggregate risk $\sum_h \omega^h(1) > \sum_h \omega^h(2)$, same beliefs: $p(1) < \pi(1)$

# Introducing Time/Sequential Trades

- Asset markets open before the realization of the uncertainty, for the trades of claims for the delivery of income (assets) contingent on the realization of each individual state.
  E.g.:
  - 
    - $q_s$: price of a claim promising the future delivery of one unit of income if and only if state $s$ is realized (contingent, or **Arrow security**), $s \in S$
- The subsequent period agents receive the net payoff on their portfolio and use their income to buy the good to be consumed (or trade again if $L > 1$ and/or $T > 1$ (dynamic economy))

# Asset Markets and Sequential Trades

Budget constraint becomes:

- for asset markets, open before realization of uncertainty:

$$\sum_s q_s \theta_s = 0 \tag{2a}$$

  where $\theta_s$ denotes, if $> 0$ (resp. $< 0$), a long (resp. short) position in contingent security $s$

- after realization of uncertainty:

$$x(s) - \omega^h(s) \leq \theta_s \text{ for each state } s = 1, ..S \tag{3}$$

- Consumer's choice problem: choose $\theta \in \mathbb{R}^S, x \in \mathbb{R}_+^S$ so as to maximize $h$'s utility (say $\sum_s \pi(s) u^h(x(s))$) subject to the two above constraints

  Note: Utility for asset holdings is only indirect

# Equilibria with sequential trades

- Market clearing (feasible allocation):
  - for assets:
  $$\sum_h \theta_s^h = 0 \text{ for each } s$$
  - for commodities:
  $$\sum_h \left(x^h(s) - \omega^h(s)\right) \leq 0 \text{ for each } s$$
- **Competitive equilibrium with sequential trades**: an array of asset prices $q$, a consumption allocation $\mathbf{x}$ and an asset allocation $\boldsymbol{\theta} = \left(.., \theta^h, ..\right)$ if
  - for all $h$, $x^h \in \mathbb{R}_+^S, \theta^h \in \mathbb{R}^S$ solve above consumer's choice problem, given $q$.
  - both the allocation of assets $\boldsymbol{\theta}$ and of consumption goods in every state $\mathbf{x}$ are feasible

# Arrow equivalence result

Theorem (Arrow 1953): The set of competitive equilibrium allocations obtained with complete markets for contingent commodities (AD equilibria) coincides with the set of competitive equilibrium allocations obtained with sequential trades and $S$ Arrow securities

## Proof.

(sketch) Let $\mathbf{p}, \mathbf{x}$ be an AD equilibrium. Set $q_s = p_s$, for all $s$. It is straightforward to verify that, with these prices, the set of budget feasible consumption bundles with the two market structures coincide, for all consumers. Hence their consumption choice will also be the same. Viceversa, let $q$ and $\mathbf{x}, \boldsymbol{\theta}$ be a competitive equilibrium with sequential trades and complete Arrow securities. Set $p_s = q_s$ for all $s$. Again it can be easily verifies that, with these prices, the set of budget feasible consumption bundles with the two market structures are the same. □

# General asset structures

- Instead of the special case of Arrow securities, suppose there are $J$ **arbitrary assets**.
  Each asset $j$ is identified by its (unit) return, a random variable $\widetilde{r_j}$, also described by its support $r_j = (.., r_j(s), ..)^T$ (e.g.: bonds, equity, derivatives,...)
- Consumer's budget constraints become:

$$\sum_s q_s \theta_s = 0 \text{ (unchanged)}$$

$$x(s) - \omega^h(s) \leq \sum_j r_j(s)\theta_j \text{ for each state } s = 1, ..S$$

# Complete vs. Incomplete Markets

- Let $R$ be the $S \times J$ matrix of the payoffs of the existing assets, with generic element $r_j(s)$
  (in the previous case, with Arrow securities, $R = I$).

- When $R$ has full rank $S$, we say **asset markets** are **complete** (for any return profile $y \in \mathbb{R}^S$, there is a portfolio $\theta$ such that $R\theta = y$)

  Equivalence result of previous page extends to any asset structure with complete markets. Hence equilibrium allocations are Pareto efficient, ...

- When rank of $R$ is less than $S$, we say **asset markets** are **incomplete**:
  In that case, sequence of budget constraints can no longer be reduced to a single intertemporal budget constraint.

# Incomplete Markets

- With incomplete markets need to study the solutions of the agents' portfolio and consumption choice problem. Its FOCs (for an interior solution) are:

$$\begin{aligned} \pi(s) Du^h(x(s)) &= \lambda_1(s) \\ \lambda_0 q &= \sum_s r(s)\lambda_1(s) = R^T \lambda_1 \end{aligned}$$

plus the budget constraints

- They imply that:
  - the MRS between the consumption between any pair of states $s, s'$, $\frac{\partial u^h(x(s))/\partial x}{\partial u^h(x(s'))/\partial x}$, is typically different across consumers $\Rightarrow$allocation of risk across consumers is (typically) not efficient

# Incomplete Markets

- *Example:* let $L = 1$, $S = 3$, $H = 2$ consumers with identical beliefs, no aggregate risk, $\omega^1(s) \neq \omega^1(s')$ for all $s \neq s'$. Suppose:

$$R = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Then at any allocation attainable with these assets we have

$$\frac{u^{1\prime}(x^1(3))}{u^{1\prime}(x^1(2))} \neq \frac{u^{2\prime}(x^2(3))}{u^{2\prime}(x^2(2))}$$

thus efficiency cannot be attained.