Microeconomics, Block I
Part 1

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Choice Theory

- **Set of alternatives**: X, with generic elements x, y
- (Weak) **Preference Relation**: $\succsim$, binary relation on X
  $x \succsim y$: x is weakly preferred to y
- **Rational Choice:** for any set $B \subseteq X$ of available alternatives, choose any element weakly preferred to any other available alternative. That is, any element in

$$C(B; \succsim) = \{x \in B : \; x \succsim y \text{ for all } y \in B\}$$

Relationship between preferences and choices:

- $\succsim$ implies choices, but also
- choices (potentially observable) allow us to make inferences over individual preferences (unobservable)

From Preferences to Choices and viceversa

- $\succsim$ **is rational** if it is:
  1. **complete:** for each pair $x, y \in X$, either $x \succsim y$ and/or $y \succsim x$ holds.
  2. **transitive:** for each triple $x, y, z \in X$,

$$x \succsim y \text{ and } y \succsim z \Rightarrow x \succsim z$$

Claim For any X finite, $C(B; \succsim)$ is ensured to be nonempty for all $B \subseteq X$ **if** $\succsim$ is rational. In contrast, we can find preferences $\succsim$ violating completeness or transitivity such that $C(B; \succsim)$ is empty. [*proof?* $\Rightarrow$ HW 1]

- If x is chosen when set of available alternatives is B, we say that, for any $y \in B$,
  $x \succsim^R y$, or x is '**revealed weakly preferred' to** y
  $\succsim^R$ is a preference relationship consistent with the observation of the choice made (and the rationality of individual)

Utility Function

- A **utility function** $U : X \to \mathbf{R}$ represents a preference relation $\succsim$ if, for all $x, y \in X$:
$$x \succsim y \;\Leftrightarrow U(x) \geq U(y)$$

Claim A preference relation can be represented by a utility function only if it is rational [*proof?*]

Note: $U(.)$ representing $\succsim$ is not unique

Sequences of choices

- Consider now a collection of sets of feasible alternatives: $\mathcal{B} = \{B_i \subseteq X, \ i \in I\}$

- Choice is now a correspondence, $C(\cdot; \succsim)$, defined for $B_i \in \mathcal{B}$: **choice rule**
  such that $C(B_i; \succsim) \subseteq B_i$ for all $B_i \in \mathcal{B}$

  What are the general properties of choice rules?

- **WARP (Weak Axiom of Revealed Preference)** is satisfied by $C(\cdot; \succsim)$ if:
  whenever we have $x, y \in B_i$ and $x \in C(B_i; \succsim)$, for some $i \in I$,
  then for any B_j such that $x, y \in B_j$ and $y \in C(B_j; \succsim)$ we must have that also $x \in C(B_j; \succsim)$

Proposition 1 If $\succsim$ is rational, then $C(\cdot;\succsim)$ must satisfy WARP.

Proof.

$x, y \in B_i$ and $x \in C(B_i;\succsim) \Rightarrow x \succsim y$
$y \in C(B_j;\succsim) \Rightarrow y \succsim z$ for all $z \in B_j$.
By transitivity, $x \succsim z$ for all $z \in B_j$ and hence whenever $x \in B_j$ we must have $x \in C(B_j;\succsim)$. □

- Can now make more specific inferences about individual preferences from observation of their choices.
- 'Inverse' result of Prop. 1 holds:

Proposition 2 If a choice rule $C(\cdot)$ on $\mathcal{B}$ satisfies WARP, there always exists a preference relation on X such that $C(B_i) = C(B_i; \succsim)$ for all $B_i \in \mathcal{B}$.

Proof.

Consider revealed preference relation $\succsim^R$ defined by $C(\cdot)$:
$x \succsim^R y$ whenever $\exists B_i \in \mathcal{B} : x, y \in B_i$ and $x \in C(B_i; \succsim)$.
It is then immediate to verify that $C(\cdot) = C(\cdot; \succsim^R)$.■ □

- Note: $\succsim^R$ may not be unique, nor rational, unless $\mathcal{B}$ is sufficiently 'rich'

Consumer

- Set of alternatives:
  consumption set, set of admissible levels of consumption of the L existing commodities.
  Will assume, as standard:

  $$X = \mathbb{R}^L_+ \text{ with generic element } x$$

  X is then a convex set, bounded below.

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- set of feasible alternatives, attainable by trading in competitive markets with a given income m:
  budget set.

Markets

- **Competitive markets:** consumer takes market prices as given, independent of his decisions (**price taker)**

In addition we consider the case where:

- *prices* are *linear*: unit price p_l of each commodity $l \in \{1, .., L\}$ is fixed, independent of level of individual trades (and the same for all agents).
- *markets are complete*: for each commodity l in X there is a market where commodity can be traded
- *free disposal* $\Rightarrow$ $p \geq 0$

Budget set:

$$B(p, m) = \left\{ x \in X : p \cdot x = \sum_{l=1}^{L} p_l x_l \leq m \right\}$$

Set of admissible alternatives is now an infinite set.

- It is convex, compact for all $p \gg 0$
- How does it change when p changes? and m?
  Note that $B(p, m) = B(\alpha p, \alpha m)$ for all $\alpha > 0$

Consumer's problem:

- choose level of trades (consumption) so as to maximize utility:

$$\max_{x} U(x)$$

$$\text{s.t. } x \in B(p, m)$$

Does a solution always exist?

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Does a solution always exist?

- Yes when:
  - $p \gg 0 \Longrightarrow B(p, m)$ is compact and
  - $U(.)$ is continuous (maintained assumption)

Continuity of U(.):

Claim: There exists a continuous utility function representing $\succsim$ if: $\succsim$ is rational and **continuous** (*upper contour set* $P(x) = \{y \in X : y \succsim x\}$ and *lower contour set* $L(x) = \{y \in X : x \succsim y\}$ are closed, for all $x \in X$)

Continuity is violated, for instance, by the following *lexicographic* preferences (for $L = 2$):

$x \succsim y$ whenever either (i) $x_1 > y_1$ or (ii) $x_1 = y_1$ and $x_2 \geq y_2$

Consumer's demand, general properties

- solution of consumer's problem for every p, m (choice rule) $\Rightarrow$ **consumer's demand** correspondence $x(p, m)$;
  is nonempty for all $p \gg 0, m \geq 0$

Will study **properties** of individual consumer's demand (restrictions on behavior implied by rationality of choice and stability of preferences) .

- $x(p, m)$ is **homogenous of degree zero** in p, m:
  $x(p, m) = x(\alpha p, \alpha m)$ for all $p \gg 0, \alpha > 0$
  (follows from property of budget set established above)

- **WARP**: let $x' \in x(p', m')$

$$\begin{aligned} \text{whenever } p \cdot x' &\leq m \text{ and } x' \notin x(p, m), \\ \text{we must have } p' \cdot x &> m' \ \forall x \in x(p, m) \end{aligned}$$

Consumer's demand, general properties, cts.

- $x(p, m)$ is a **upper hemicontinuous** (correspondence, as solution may not be unique) in p, m, for all $p \gg 0, m > 0$

- Why?

Consumer's demand, general properties, cts.

- $x(p, m)$ is a **upper hemicontinuous** (correspondence, as solution may not be unique) in p, m, for all $p \gg 0, m > 0$

- Why?
- Follows from the Theorem of the Maximum (by the continuity of $U(.)$ and of the budget set correspondence $B(p, m)$).

Assumptions on Preferences and Properties of Demand

For other properties, may impose (some) additional assumptions:

- **A.1 local non satiation**: $\succsim$ is such that, for all $x \in X$ there exists $y \in N(x) \cap X$, an arbitrarily small neighborhood in X of x, such that $y \succ x$ (that is, $x \not\succsim y$).

  implies that indifference curves are not 'thick'.
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- **Claim A.1:** Under A.1 , $x(p, m)$
  satisfies the budget identity: $p \cdot x(p, m) = m$ for all p, m
  [*Proof?*]

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  A.2' strict convexity: if, in addition, $y \sim x \Rightarrow \alpha x + (1-\alpha)y \succ x$ for all $x, y \in X, \alpha > 0$
  (that is, indifference curves have no 'flat parts'; corresponds to strict quasi-concavity of $U(.)$)

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- **Claim A.2:**
  Under A.2, $x(p, m)$ is convex valued
  Under A.2' $x(p, m)$ is single valued (a function)
  [*Proof?*]

Assumptions on Preferences and Properties of Demand cts.

- **A.3 homothetic**: preferences are invariant with respect to expansions along rays from the origin, $x \sim y \Rightarrow \alpha x \sim \alpha y$ for all $x, y \in X$, $\alpha > 0$

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- **Claim A.3:** Under A.3, $x(p, \alpha m) = \alpha x(p, m)$ for all $m, p \gg 0, \alpha > 0$ [*Proof? in HW1*]

'Standard' utility functions:

- Cobb Douglas $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$
- CES: $u(x_1, x_2) = \frac{1}{\sigma} x_1^{\sigma} + \frac{1}{\sigma} x_2^{\sigma}$,
- quasi linear: $u(x_1, x_2) = x_1 + v(x_2)$, with $v' > 0, v'' < 0$
- linear: $u(x_1, x_2) = \alpha x_1 + \beta x_2$
- Leontief: $u(x_1, x_2) = \min(\alpha x_1, x_2)$

- Do they satisfy A.1, A.2 and A.3? under what conditions?

FOCs

Whenever $U(.)$ is continuously differentiable, any solution of the consumer's problem satisfies the following system of Kuhn Tucker necessary conditions,
stated here for the case of interior solutions, assuming A.1 (Ins) holds:

$$\frac{\partial U}{\partial x_l} = \lambda p_l, \; l = 1, .., L$$

or, more compactly,

$$DU - \lambda p = 0$$

and

$$m - p \cdot x = 0.$$

for some $\lambda > 0$ (Lagrange multiplier).

If $A.2'$ ($U(.)$ strictly quasi concave) also holds, these conditions are also sufficient and solution is unique.

Comparative statics

- **Claim:** Assume $U(.)$ is twice continuously differentiable and satisfies $A.1$, $A.2'$: $x(p, m)$ is then a continuously differentiable function at all p, m such that $x(p, m) \gg 0$.

Under above conditions, properties of $x(p, m)$ can be obtained with IFT from previous FOCs:

$$\begin{bmatrix} D^2U & -p \\ -p^T & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda I \\ x^T \end{bmatrix} dp + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dm$$

- **Claim:** Under above assumptions,

$$\begin{bmatrix} D^2U & -p \\ -p^T & 0 \end{bmatrix} \text{ is invertible}$$

(when evaluated at (x, λ, p) satisfying FOCs).

Proof: Suppose not; i.e. $\exists (y^T, z) \in \mathbb{R}^{L+1}$, $(y^T, z) \neq 0$:

$$\begin{bmatrix} D^2U & -p \\ -p^T & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} D^2Uy - pz \\ -p \cdot y \end{bmatrix} = 0$$

Premultiplying $D^2Uy - pz$ by y^T yields:

$$y^T D^2Uy - y \cdot pz < 0 \text{ if } y \cdot p = 0 \text{ (by A.2}'\text{)},$$

a contradiction. ■

Comparative statics cts.

Setting:

$$\begin{bmatrix} D^2U & -p \\ -p^T & 0 \end{bmatrix}^{-1} \equiv \begin{bmatrix} S & -v \\ -v^T & t \end{bmatrix}$$

we get:

$$D_m x = v$$

$$D_p x = \lambda S - vx^T = \lambda S - D_m x \; x^T$$

where the following properties hold:

$$v \cdot p = 1$$

S is symmetric, negative semi-definite and such that $p^T S = 0$.

[*Proof? in HW2*]

Comparative statics cts.

Will show the properties found above of $D_p x$, $D_m x$ depend on the properties of demand we had already seen:

- The homogeneity property of $x(p, m)$ implies:

$$D_p x \; p + D_m x \; m = 0$$

- Claim A.1 (budget identity property).then implies:

$$\begin{cases} p^T D_p x + x^T = 0^T \\ p \cdot D_m x = 1 \end{cases}$$

  [Check these properties are satisfied by expressions found in previous slide]

- WARP $\Longrightarrow$? [more later]

Comparative statics (homothetic preferences)

If A.3 also holds, from Claim A.3 we get:

$$D_m x = x(p, 1) >> 0$$

Indirect Utility

$$V(p, m) = U(x(p, m))$$

Properties:

- $\partial V(p, m)/\partial m = \lambda > 0$: consumer's marginal utility of wealth equals the shadow value of relaxing the constraint
  $\partial V(p, m)/\partial p_l \leq 0$ for all l, p
  [*argument:*?]

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  $\partial V(p, m)/\partial p_l \leq 0$ for all l, p
  [*argument:*?]
- $V(p, m)$ is quasi-convex in p: the lower contour set $\{p : V(p, m) \leq \bar{V}\}$ is convex

  [*argument:* Take any pair p', p'' such that $V(p', m), V(p'', m) \leq \bar{V}$, and consider $\hat{p} = \alpha p' + (1 - \alpha)p''$ for $\alpha \in [0, 1]$. Note that for all x such that $\hat{p} \cdot x \leq m$ we must have either $p' \cdot x \leq m$ and/or $p'' \cdot x \leq m$; thus $U(x) \leq \bar{V}$.]

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  - Another, more obvious property is that $V(p, m)$ is homogenous of degree zero in p, m

Duality

Consider the problem:

$$\min_x p \cdot x$$

$$\text{s.t. } U(x) \geq u$$

A solution again exists for all $u \geq U(0)$ and $p \gg 0$ (and may now exist even when $p \geq 0$ under strict monotonicity).

Let us denote its solution for all p, u by $x(p, u)$ (sometimes referred to as **compensated** - or Hicksian - **demand**)
and $E(p, u) = p \cdot x(p, u)$ define the **expenditure function**.

Duality cts.

Claim Under A.1, we can show that, for all $p \gg 0$, $m > 0, \; u > U(0)$, the following identities hold:

$$\begin{aligned} x(p,m) = x(p,u^*) \text{ for } u^* &= V(p,m) \\ \text{and } E(p,u^*) &= m \\ x(p,u) = x(p,m^*) \text{ for } m^* &= E(p,u) \\ \text{and } V(p,m^*) &= u \end{aligned} \tag{1}$$

[*argument:*

$$\text{if } x \in x(p,u^*), \text{ then } u(x') \geq u(x) \Rightarrow p \cdot x' \geq p \cdot x;$$
$$\text{if } x \in x(p,m^*), \text{ then } u(x') > u(x) \Rightarrow p \cdot x' > p \cdot x.$$

Use then A.1.]

Slutsky Equation

- Using previous relationship, from FOCs of the compensated demand problem

$$\begin{cases} -p + \mu DU = 0 \\ U(x) - \bar{u} = 0 \end{cases}$$

we get (for $\bar{u} = u^* = V(p, m)$):

$$D_p x(p, u) = \lambda S$$

hence symmetric, negative semi-definite.
[*proof? see next page*]

- Hence, for $u = V(p, m)$:

$$D_p x(p, m) = D_p x(p, u) - D_m x \, x^T$$

Note: this property can also be obtained by differentiating the second identity in (1) above, after substituting $E(p, u)$ for m^*.

Argument for claim $D_p x(p,u) = \lambda S$:
Applying again the IFT to the system of FOC's in previous page, evaluated at $\bar{u} = u^* = V(p,m)$:

$$\begin{bmatrix} \mu D^2 U & DU \\ DU^T & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\mu \end{bmatrix} = \begin{bmatrix} I \\ 0^T \end{bmatrix} dp + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\bar{u}$$

Note next, by comparing FOC's of the primal and dual problem when the solution for x is the same and so is p, that $\mu = 1/\lambda$. It is then possible to verify [*e.g. by using formulae for partitioned inverse*] that

$$\begin{bmatrix} \mu D^2 U & DU \\ DU^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda S & b \\ b^T & z \end{bmatrix}$$

for some b, z.■

Expenditure function

Properties of the expenditure function:

- $\partial E(p, u)/\partial u > 0$ (under Ins)
- $\partial E(p, u)/\partial p_l \geq 0$ for all $l = 1, .., L$
- $E(p, u)$ is concave in p

  [*argument:* For any pair p', p'', consider $\hat{p} = \alpha p' + (1-\alpha)p''$ for $\alpha \in [0, 1]$. Then, for any u, $E(\hat{p}, u) = \hat{p} \cdot x(\hat{p}, u) = \alpha p' \cdot x(\hat{p}, u) + (1-\alpha)p'' \cdot x(\hat{p}, u)$, which is in turn $\geq \alpha E(p', u) + (1-\alpha)E(p'', u)$.]

Also, $E(p, u)$ is homogeneous of degree one in p

From expenditure and indirect utility functions to demand

- By the envelope theorem, compensated demand can be obtained from expenditure function:

$$x(p,u) = D_p E(p,u)$$

hence the properties of $x(p,u)$ can also be obtained from those of $E(p,u)$ [e.g., that $D_p x(p,u)$ is negative semi-definite]

- Similarly, differentiating the equation defining the indirect utility, $V(p,m) = U(x(p,m))$, wrt p and using FOCs of the consumer's problem and the property shown above (p.23) $p^T \; D_p x = -x^T$, yields [**Roy's identity**]:

$$D_p V(p,m) = -\lambda x(p,m) = -\frac{\partial V(p,m)}{\partial m} x(p,m)$$

WARP again

WARP for consumer's choice problem implies:

$$\text{if } p \cdot x(p', m') = m \text{ and } x(p', m') \neq x(p, m), \text{ then } p' \cdot x(p, m) > m'$$

- Can be restated as:

  for any *compensated price change* $(p', m') \rightarrow (p, m = p \cdot x(p', m'))$ we must have $(p - p') \cdot (x(p, m) - x(p', m')) = \Delta p \cdot \Delta x \leq 0$ (weak inequality because Δx may be 0)

- **Claim:** Take an arbitrary function $x(p, m)$ that is homogenous of degree zero, differentiable and satisfies WARP as well as the budget identity. Then the matrix of compensated price effects

$$D_p x(p, m) + D_m x \, x^T \text{ is negative semidefinite.}$$

WARP again

[*argument*: differential version of compensated price change:

$$dp, dm: \; dm = x \cdot dp$$

Induced demand change (change in compensated demand) is

$$dx = D_p x \; dp + D_m x(x \cdot dp)$$

and we must have, by WARP

$$dp \cdot dx \leq 0,$$

and this is true for any dp]

- Note that now WARP is not enough to ensure (such matrix is also symmetric and hence) that we can find preferences rationalizing demand $x(p, m)$

Producer Theory

Focus on production activity of firms operating in competitive markets

- production plan: $y \in \mathbb{R}^L$, net output of the L goods

$$\begin{aligned} y_i &< 0 : \text{ input} \\ y_i &> 0 : \text{ output} \end{aligned}$$

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- Set of alternatives: **production set**, set of (technologically feasible) production plans

$$Y \subset \mathbb{R}^L$$

  assumed to satisfy the following standard *properties*:
  - nonempty
  - closed
  - $Y \cap \mathbb{R}^L_+ = \{0\}$: no free lunch and possibility of inaction
  - free disposal: $y \in Y$ and $y' \leq y \Rightarrow y' \in Y$

Production function

Whenever the commodities which are outputs are fixed, $O \subset \{1, .., L\}$ (and hence also the complementary set of those which are inputs), the outer boundary of Y can typically be represented by a (continuous) function:

- the **production function**, describing the maximal output level attainable for any level of inputs, written here for the case where $O = \{1\}$:

$$y_1 = f(z) \text{ iff}$$
$$\text{(i) } (y_1, -z) \in Y$$
$$\text{(ii) } \nexists y_1' > y_1 : \ (y_1', -z) \in Y$$

$f : \mathbb{R}_+^{L-1} \rightarrow \mathbb{R}_+$ (weakly) monotonically increasing by the free disposal property

Firm's problem:

choose production plan so as to maximize firm's profits
(*why? more later*):

$$\max \pi = p \cdot y$$

$$s.t.\ y \in Y$$

$$[\text{or } \max \pi = p_1 f(z) - w \cdot z, \text{ for } w \equiv (p_2, .., p_L)]$$

- Existence of a solution requires now additional conditions (feasible set may not be compact _ analogies with expenditure minimization problem?).

- Solution for every $p \geq 0$: **firm's net supply** correspondence $y(p)$

  Value of the solution: **profit function** $\pi(p) = p \cdot y(p)$

Properties of Net Supply correspondence and Profit function

- $y(p)$ - if nonempty - is homogeneous of degree 0 in p
- $\pi(p)$ is homogeneous of degree 1 in p
- $\pi(p)$ is a convex function (contrast with $E(p,u)$, and $V(p,m)$] [*argument:* take any pair p', p'' and consider $\hat{p} = \alpha p' + (1-\alpha)p''$ for $\alpha \in (0,1)$. Note that $\pi(\hat{p}) = y(\hat{p}) \cdot (\alpha p' + (1-\alpha)p'') \leq \alpha y(p') \cdot p' + (1-\alpha)y(p'') \cdot p'' = \alpha\pi(p') + (1-\alpha)\pi(p'')$.]

- **B.1 Convexity:** Y is convex.
  This implies:
  (i) not only that, analogously to consumer, 'more balanced' combinations of inputs are more productive;
  (ii) but also that returns to scale are non increasing: reducing scale does not decrease productivity,

$$y \in Y \Rightarrow \alpha y \in Y \text{ for all } \alpha \in [0, 1]$$

Corresponding property of $f(.)$:

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  (ii) but also that returns to scale are non increasing: reducing scale does not decrease productivity,

$$y \in Y \Rightarrow \alpha y \in Y \text{ for all } \alpha \in [0, 1]$$

Corresponding property of $f(.)$:

- $f(.)$ is concave.

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  Corresponding property of $f(.)$:
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- **Claim B.1:** Under B.1, $y(p)$ is convex-valued [*why?*]

Assumptions on Technology and other Properties of Supply

Strengthening B.1:

- **B.1' Strict Convexity:** $y, y' \in Y \Rightarrow \alpha y + (1-\alpha)y' \in intY$
  Corresponds to strict concavity of $f(.)$.
  Implies that: $y(p)$ is single valued for all $p \gg 0$

  Alternatively:

Assumptions on Technology and other Properties of Supply

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  Alternatively:

- **B.1" Convex Cone:** $y \in Y \Rightarrow \alpha y \in Y$ for all $\alpha > 0$
  returns to scale are constant (productivity invariant to scale)

  Corresponds to homogeneity of degree 1 of $f(.)$.
  Implies that:
  \- $y(p)$ is not single valued: $y \in y(p) \Rightarrow \alpha y \in y(p)$ for all $\alpha > 0$
  \- $\pi(p) = 0$ for all $p : y(p)$ is nonempty.

FOCs

- Whenever $f(.)$ is differentiable, any solution of the firm's problem satisfy the following system of first order conditions
(stated here for the case of interior solutions - (locally) unconstrained problem):

$$p_1 Df = w$$

also sufficient condition if $f(.)$ is concave.

- If $f(.)$ is strictly concave, $y(p)$ is a continuous function.

Comparative statics

Assume $f(.)$ is twice continuously differentiable and strictly concave: again applying IFT to FOCs in previous slide, we get:

$$D_w z = \frac{1}{p_1}\left(D^2 f\right)^{-1} \text{ symmetric, negative definite}$$

- From profit to net supply functions:

$$y(p) = D_p \pi \text{ (by the envelope theorem),}$$

so that $D_p y = D_p^2 \pi$ is:

  - symmetric,
  - *positive* semi-definite (by the convexity of $\pi(.)$) - recall $z = -(y_2, .., y_L)$! - and
  - such that $D_p y \; p = 0$ (by the homogeneity of $y(p)$).

Cost Minimization

The input level z which solves the firm's choice problem also solves the following problem:

$$\min C = w \cdot z$$

$$s.t.\ f(z) \geq y_1$$

- This is perfectly analogous to expenditure minimization problem of consumer. Hence we know that:
  - $C(w, y_1)$ is concave in w and such that $\partial C/\partial y_1 > 0$ (if $f(.)$ is strictly increasing) and $\partial C/\partial w_l \geq 0$, $l = 2, .., L$
  - $z(w, y_1) = D_w C$ exhibits same properties of compensated demand function (see previous section: $D_w z(w, y_1)$ negative semi-definite, ..).

Why Profit Maximization?

- Firm is owned by a (set I of) consumers
- Ownership of a fraction of the firm for consumer i means the right to receive a fraction $\theta_i \in [0, 1]$ of the firm's profits.
- When markets are competitive, the choice of alternative production plans by the firm has the following effect on the consumer's choice problem:

$$\begin{aligned} & \max_x U(x) \\ \text{s.t. } p \cdot x \quad \leq & \quad m + \theta_i (p \cdot y) \end{aligned}$$

Thus firm's choices only affect consumer by modifying his income. Hence the consumer favors the choice of the production plan that maximizes the firm's profits $p \cdot y$

Aggregate Demand

- Aggregate demand with H consumers:

$$x(p, \left(m^h\right)_{h=1}^H) = \sum_{h=1}^{H} x^h(p, m^h)$$

  - What can we say about its properties?

- It is clearly:
  - continuous (if so are individual demands),
  - homogeneous of degree 0 in $(p, \left(m^h\right)_{h=1}^H)$,
  - satisfies Walras law:

$$p \cdot x(p, \left(m^h\right)_{h=1}^H) \leq \sum_h m^h \text{ for all } p, (m^h)_{h=1}^H$$

  - its Jacobian has less 'structure' the larger is H [*why?*]:

$$D_p x(p, \left(m^h\right)_{h=1}^H) = \sum_h \left[\lambda^h S^h - v^h \left(x^h\right)^T\right]$$

Aggregation: no distribution effects

1. When does aggregate demand depend only on aggregate income $\mathbf{m} \equiv \sum_h m^h$, not on its distribution?
   Aggregate demand is invariant:

   (i) *wrt any infinitesimal change in the distribution* of income $\left(dm^h\right)_{h=1}^{H} : \sum_h dm^h = 0$:
   if, and only if

   $$D_m x^h(p, m^h) = D_m x^{h'}(p, m^{h'}) \text{ - that is } v^h = v^{h'} \text{ - for any pair } h, h'$$

   (ii) *wrt any (also discrete) change in the distribution* of income $\left(\Delta m^h\right)_{h=1}^{H} : \sum_h \Delta m^h = 0$:
   when, in addition, $D_m x^h(p, m^h)$ is independent of m^h, for all h.
   That is, consumers have identical homothetic preferences so that $x^h(p, m^h) = x(p, 1)m^h$ for all h.

2. When does aggregate demand exhibit the same properties of individual demand?

   We say a *representative consumer* exists when there is a utility function $\mathbf{U} : \mathbb{R}^L_+ \to \mathbb{R}$ such that:

$$\begin{aligned} x\left(p, \left(m^h\right)_{h=1}^H\right) &= x(p, \mathbf{m}; \mathbf{U}) \\ \text{and } x(p, \mathbf{m}; \mathbf{U}) &\in \arg\max \begin{cases} \mathbf{U}(x) \\ \text{s.t. } p \cdot x \leq \mathbf{m} \end{cases} \end{aligned}$$

It exists:

(i) always *at a given point* $(\bar{p}, (\bar{m}^h)_{h=1}^H)$ (that is, at $(\bar{p}, (\bar{m}^h)_{h=1}^H)$, $\exists \mathbf{U}$: $x(\bar{p}, (\bar{m}^h)_{h=1}^H) = x(\bar{p}, \mathbf{m}; \mathbf{U})$).
Let:

$$\mathbf{U}(x, \left(\xi^h\right)_{h=1}^H) = \begin{cases} \max \sum_h \xi^h U^h(x^h) \\ \text{s.t } \sum_h x^h \leq x \end{cases} \tag{2}$$

It is easy to verify that if we set $\xi^h = \lambda / \left(\bar{\lambda}^h\right)$ for all h, for some $\lambda > 0$, we have $x(\bar{p}, (\bar{m}^h)_{h=1}^H) \in \arg\max_{p \cdot x \leq \mathbf{m}} \mathbf{U}(x, \left(\xi^h\right)_h)$.
[*argument:* Let $\bar{x}^h = x^h(\bar{p}, \bar{m}^h)$, $\bar{x} = \sum_h \bar{x}^h$. We need to show that $D\mathbf{U}(\bar{x}) = \lambda \bar{p}$ for some $\lambda > 0$, knowing that $DU^h(\bar{x}^h) = \bar{\lambda}^h \bar{p}$ for all h.
Note that, at a solution of above problem (2), for all x we have $D\mathbf{U}(x) = \xi^h DU^h(x^h)$ for all h. [*why?*]
Premultiplying $DU^h(\bar{x}^h) = \bar{\lambda}^h \bar{p}$ by ξ^h yields: $\xi^h DU^h(\bar{x}^h) = \xi^h \bar{\lambda}^h \bar{p}$ for all h. Using $\xi^h = \lambda / \left(\lambda^h\right)$, we obtain that $\xi^h DU^h(\bar{x}^h)$ is h-invariant, and hence equal to $D\mathbf{U}(\bar{x})$, and this is in turn equal to $\lambda \bar{p}$.]

(ii) *locally* (that is, at a given point $\left(p, \left(m^h\right)_{h=1}^H\right)$ and wrt infinitesimal changes in p):

at $\left(p, \left(m^h\right)_{h=1}^H\right)$ $\exists \mathbf{U}$:
$x\left(p, \left(m^h\right)_{h=1}^H\right) = x(p, \mathbf{m}; \mathbf{U}))$
and $D_p x\left(p, \left(m^h\right)_{h=1}^H\right) = D_p x(p, \mathbf{m}; \mathbf{U}))$.

The latter property requires:

$$D_p x\left(p, \left(m^h\right)_{h=1}^H\right) = \boldsymbol{\lambda}\mathbf{S} - \mathbf{v}\mathbf{x}^T$$

for some $\boldsymbol{\lambda} > 0$, $\mathbf{S}$ symmetric, negative semidefinite, of rank $L-1$, such that $p^T\mathbf{S} = \mathbf{0}$, and $\mathbf{v}$ such that $p \cdot \mathbf{v} = 1$.
This holds if $v^h = \mathbf{v}$ for all h (analogies with 1(i)).

(iii) *globally*:

$$\exists \mathbf{U} : x\left(p, \left(m^h\right)_{h=1}^H\right) = x(p, \mathbf{m}; \mathbf{U}) \text{ for all } p, \left(m^h\right)_{h=1}^H.$$

Property holds if consumers have identical homothetic preferences. In this case $\mathbf{U}(x) = U^h(x)$ for all h and $\mathbf{m} = \sum_h m^h$.

- And if all consumers have quasilinear preferences?
- Representative consumer may exist, under weaker conditions on preferences, if we *fix the distribution of income*.

Law of Demand and WARP

- In general, as shown above (1.), aggregate demand depends on income distribution. We can still write it as a function of aggregate income $\mathbf{m}$ if we fix the distribution of income:

$$m^h = \alpha^h \mathbf{m} \text{ for some given } \alpha^h \geq 0, \text{ for all } h, \ \sum_h \alpha^h = 1.$$

- Does aggregate demand $x(p, \mathbf{m})$ satisfy WARP?
  In general no.

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- Does aggregate demand $x(p, \mathbf{m})$ satisfy WARP?
  In general no.
    - Take $(p, \mathbf{m})$ and $(p', \mathbf{m}')$ such that $x(p', \mathbf{m}') \neq x(p, \mathbf{m})$ and:

    $$p \cdot x\left(p', \mathbf{m}'\right) \leq \mathbf{m}.$$

    It is immediate to see that the following inequality can still hold:

    $$p' \cdot x(p, \mathbf{m}) \leq \mathbf{m}'$$

    [because $p \cdot x(p', \mathbf{m}') \leq \mathbf{m}$ does not imply $p \cdot x^h\left(p', m^{h\prime}\right) \leq m^h$ for all h; similarly for $p' \cdot x(p, \mathbf{m}) \leq \mathbf{m}'$.]

Law of Demand and WARP cts.

- A sufficient condition for aggregate demand to satisfy WARP is that individual (uncompensated) demand satisfies the **Law of Demand** (LOD)**:**

  $(p' - p) \cdot \left(x(p', \mathbf{m}) - x(p, \mathbf{m})\right) < 0$ for all $p, p', \mathbf{m}$**:** $x(p', \mathbf{m}) \neq x(p, \mathbf{m})$

  [This implies that $D_p x^h$ is negative semidefinite for all h, always true for compensated demand (that is, substitution effects prevail over income effects).]

  This property carries over to aggregate demand and we show it implies WARP.
  [Let $(p, \mathbf{m}), (p', \mathbf{m}')$: $x(p, \mathbf{m}) \neq \mathbf{x}(p', \mathbf{m}')$, $p \cdot x(p', \mathbf{m}') \leq \mathbf{m}$. Take $p'' \equiv (\mathbf{m}/\mathbf{m}')p',$; $x(p'', \mathbf{m}) = x(p', \mathbf{m}')$ by the homogeneity of degree zero of demand. By LOD $(p'' - p) \cdot [x(p'', \mathbf{m}) - x(p, \mathbf{m})] < 0$. Using Walras law and the property $p \cdot x(p'', \mathbf{m}) \leq \mathbf{m}$ we then get $p'' \cdot x(p, \mathbf{m}) > \mathbf{m} \Leftrightarrow . p' \cdot x(p, \mathbf{m}) > \mathbf{m}'$. QED]

- A sufficient condition for individual demand to satisfy the Law of Demand is that preferences of every agent are homothetic (though possibly different among them). [*proof?*]

Aggregation and Production

- Suppose there are F firms, with production set Y^f, $f = 1, .., F$.
  When does a *representative producer* exist?
  That is, when can we find a production set $\mathbf{Y}$ such that

$$y(p) \in \arg\max_{y \in \mathbf{Y}} p \cdot y = \sum_f p \cdot y^f(p) \ ?$$

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- Always, no restriction needed in this case:

$$\text{set } \mathbf{Y} = \sum_f Y^f$$

  [*proof? see HW3*]

  Note also that

$$\sum_f D_w z^f = \frac{1}{p_1} \sum_f \left(D^2 f^f \right)^{-1}$$

  which is also always symmetric, negative definite (LOD).