When Are Asymmetric Information Economies Walrasian ? A Survey

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1 Introduction

After the initial contributions of Radner (1968) and Prescott-Townsend (1984), the analysis of competitive equilibria of economies with asymmetric information has recently received renewed attention. For such economies the interaction between the private information dimension (e.g., the unobservable action in the moral hazard case, the unobservable type in the adverse selection case) and the observability of agents' trades plays a crucial role, since trades have typically informational content over the agents' private information. In particular, to decentralize incentive efficient Pareto optimal allocations the availability of fully exclusive contracts, i.e. of contracts whose terms (price and payoff) depend on the transactions in all other markets of the agent trading the contract, is generally required. The implementation of these contracts imposes typically the very strong informational requirement that all trades of an agent need to be observed. It is then of interest to analyze also situations where contracts traded are necessarily non-exclusive, because perfect monitoring of trades is not available. The case of complete anonymity of trades, where no transaction of the agents is observable, constitutes an important benchmark in this respect.

In summary, an important dichotomy arises in the study of economies with asymmetric information: economies in which each agent's trades are observable behave very differently from economies in which trades are not observable (often these economies are referred to, respectively, as economies with exclusive and non-exclusive contractual relationships).

Many different approaches have been taken in the literature to analyze equilibria of economies with asymmetric information. Even restricting to Walrasian equilibria, many different definitions are available (a situation that Douglas Gale has referred to as "Balkanization"). In this paper we attempt at an analysis of such concepts. To facilitate comparisons we apply all such concepts to the same simple moral hazard and adverse selection economy.

We will show that the equilibrium concepts developed for moral hazard economies have an analogous applications to adverse selection economies, and viceversa. We will also show that, in economies with observable trades, for which different concepts have been developed, all such concepts generate coincident predictions in terms of equilibrium allocations and prices. In other words, different equilibrium concepts give rise to different equilibrium predictions only when they capture different assumptions about the observability of trades. All this for both moral hazard and adverse selection economies, in our simple example economies.

Our classification of equilibrium concepts follows. In the framework of a Walrasian competitive equilibrium model, alternative assumptions on the observability of agents' trades may be captured in a reduced form by alternative assumptions on the possible non-linearities of equilibrium prices.

Complete anonymity of trades (full non-exclusivity) corresponds to restricting price schedules to be a linear function of trades. The intermediate case in which only short and long trading positions can be distinguished, will turn out to be central in our analysis: a minimal form of non-linearity, e.g., the possibility of having a different price for buyers and sellers (a bid-ask spread), is in fact necessary and sufficient for competitive equilibria to exist; see Dubey, Geanakoplos and Shubik (1995), Bisin and Gottardi (1998), Bisin, Geanakoplos, Gottardi, Minelli and Polemarchakis (1998).

At the other extreme, complete observability of trades (exclusivity) is captured by allowing price schedules to be general non-linear function of agents' trades. We distinguish two main approaches to the analysis of such economies:

Prices are arbitrarily non-linear maps; as a consequence minimal restrictions are imposed by the equilibrium notion, and hence the plethora of resulting equilibria is refined by a formal concept in the spirit of sequential equilibria; see Gale (1993), Dubey Geanakoplos-Shubik (1995), Lisboa (1996), Magill-Quinzii (1998).

The specification of agents' budget sets restricts admissible trades to lie in the set of incentive compatible trades; in other words, non-incentive compatible trades are just non available for trade, or, say, are traded at infinite price; Prescott-Townsend (1984), see also Bisin-Gottardi (2000).

But when is a Walrasian equilibrium concept justified ? First of all, contracts must be written on purely idiosyncratic uncertainty; thus agents, though not informationally small, are 'small' as far as the level of their trades is concerned, so that their price-taking behavior is appropriate.

To better evaluate the informational requirements and the structure of markets implicit in these Walrasian equilibrium notions, it is important though to examine the conditions under which such equilibria can be obtained as the limit, as the number of strategic traders gets large, of the Nash equilibria of a game (where information and strategy sets are explicitly modelled).

But convergence to which definition of Walrasian equilibrium ? Several such definitions exist, as we argued above: even conditioning on the available information regarding agents' trades, we identified two main Walrasian equilibrium concepts for economies with observable trades.

Also, convergence of which market game ? For instance, many different strategic equilibrium concepts have been introduced to study adverse selection economies with exclusive contracts. The standard strategic analysis of such economies, due to Rothschild-Stiglitz (1976), considers the Nash equilibria of a game in which insurance companies simultaneously choose the contracts they issue, and the competitive aspect of the market is captured by allowing the free entry of insurance companies. Such equilibrium concept does not perform too well: equilibria in pure strategies do not exist for robust examples (Rothschild-Stiglitz (1976)), while equilibria in mixed strategies exist (Dasgupta-Maskin (1986)) but, in this set-up, are of difficult interpretation. Even when equilibria in pure strategies do exist, it is not clear that the way the game is modelled is appropriate for such markets, since it does not allow for dynamic reactions to new contract offers (Wilson (1977) and Riley (1979); see also Maskin-Tirole (1992)). Moreover, once sequences of moves are allowed, equilibria are not robust to 'minor' perturbations of the extensive form of the game (Hellwig (1987)).

The literature on the strategic analysis of economies of asymmetric information presents us with too many equilibrium concepts and strategic game forms, and too few robust predictions about equilibrium allocations to be of much help at this stage. This paper studies instead Walrasian equilibrium concepts, with the objective of identifying robust predictions in terms of equilibrium allocations. The aim is to provide therefore a reference point for the convergence analysis. Perhaps if we have a good sense of where strategic equilibria should converge to, we can identify strategic game forms whose equilibria actually do converge.[1]

2 The Economy: Moral Hazard and Adverse Selection

We study two simple economies, workhorses of economics of uncertainty. The first economy is characterized by moral hazard in the form of hidden action; see Arrow (??), Grossman-Hart (??). The second economy is characterized by

[1]Not much work exists on convergence to competitive equilibria in economies with asymmetric information. Bisin-Gottardi-Guaitoli (1998), for a simple moral hazard (hidden action) economy show that, with complete observability of trades convergence holds; on the other hand, when information over agents' trades is more limited, convergence is not always ensured. For a particular class of economies with adverse selection, Biais-Martimort-Rochet (1997) obtain convergence when intermediaries can prevent agents from buying multiple units of the contracts they issue.

adverse selection in the form of unobservable risk types; see Rothschild-Stiglitz (1977), Wilson (19??). We will introduce such economies in a as much unified way as possible.

There is measure 1 of agents who live two periods, $t = 0, 1$, and consume, only in period 1, a single consumption good. Uncertainty is purely idiosyncratic, and is described by the collection of random variables $\tilde{s}^\tau$, $\tau \in [0, 1]$, assumed to be identically and independently distributed, with support $S = \{H, L\}$; the realization of all $\tilde{s}^\tau$ variables is commonly observable[2] Uncertainty enters the economy via the agents' endowments. The (date 1) endowment of an agent $\tilde{w}^\tau = w(\tilde{s}^\tau)$; let $w^H \equiv w(H), w^L \equiv w(L)$ be the agent's endowment in, respectively, the idiosyncratic state H and state L.

The probability distribution of the period 1 endowment that each agent faces depends from the value taken by a variable $e \in \{h, l\}$. The interpretation of e is what distinguishes moral hazard from adverse selection economies.

In moral hazard economies, e is an unobservable level of effort which is chosen by the agent. In adverse selection economies e describes the exogenously given risk type of an agent, and its realization is only privately observable. Let ξ^h (resp. $\xi^l = 1 - \xi^l$) be the probability that an agent is of type $e = h$ (resp. $e = l$); by the Law of Large Numbers, ξ^h is then also the fraction of agents in the population which are of type h. Importantly, in adverse selection economies, all markets open after agents observe the probability distribution of their endowments.

Let π_s^e be the probability of the realization s given $e \in \{h, l\}$ (obviously $\pi_H^e = 1 - \pi_L^e$, for any e). By the Law of Large Numbers, π_s^e is also the fraction of agents with e for which state s is realized. Agents' preferences are represented by a (Von Neumann - Morgenstern) utility function of the following form:

$$\pi_s^e u(c^H) + (1 - \pi_s^e) u(c^L) - v(e)$$

where (c^H, c^L) denotes consumption respectively in state H and L; let $c \equiv (c^H, c^L)$, $w \equiv (w^H, w^L)$.

In moral hazard economies, $v(e)$ denotes the disutility of effort e, and we assume that

$$\pi_H^h > \pi_H^l, \quad v(h) > v(l), \quad w^H > w^L > 0$$

so that h is the 'high' effort and H is the 'good' state.

In adverse selection economies, $v(e)$ is just a utility constant and hence is disregarded from the analysis without loss of generality.

Assumption 1 *Preferences are strictly monotonic, strictly concave, twice continuously differentiable, and* $\lim_{x \to 0} u'(x) = \infty$.

[2] Measurability issues arise in probability spaces with a continuum of indipendent random variables arise; see ??.

Let Ω be the set of parameter values $(v(h), v(l), \pi_H^h, \pi_H^l, w^H, w^L)$ of the economy which satisfy the above assumptions.

3 The Symmetric Information Benchmark

We consider now the benchmark case of symmetric information, in which e, be it endogenous effort or risk type, is commonly observed.

Definition 1 *An allocation* $(\bar{c}, \bar{e}) \in \Re_+^2 \times \{h, l\}$ *of consumption and effort is* optimal *in the moral hazard economy under symmetric information if it solves:*

$$max_{c,e} \sum_s \pi_s^e u(c_s) - v(e) \tag{1}$$

s.t.

$$\sum_s \pi_s^e (c_s - w_s) = 0$$

Definition 2 *An allocation* $(\bar{c}^h, \bar{c}^l) \in \Re_+^4$ *of consumption is* optimal *in the adverse selection economy under symmetric information if it solves:*

$$max_{c^h, c^l} \sum_e \kappa^e \sum_s \pi_s^e u(c_s^e) \tag{2}$$

s.t.

$$\sum_e \xi^e \sum_s \pi_s^e (c_s^e - w_s) = 0$$

for some $(\kappa^h, \kappa^l) \gg 0$ *such that* $\kappa^h = 1 - \kappa^l$*.*

Let q_s^e denote the (linear) price of consumption in state s for agents e. By allowing the prices of the securities whose payoff is contingent on the idiosyncratic uncertainty to depend on e, we effectively allow agents to trade in a complete set of markets.

In addition to consumers we introduce firms. Firms 'pool' payments in different states of the world. The Law of Large Numbers provides, in the economy under consideration, a mechanism - or a technology - for transforming aggregates of the commodity contingent on different individual states. Thus firms are characterized by the following constant returns to scale technology:

$$Y = \{y \in \Re^4 : \sum_e \sum_s \pi_s^e y_s^e \leq 0\}$$

where $y \equiv [y_s^e]_{s \in S}^{e \in}$. :

The firms' problem is then the choice of a vector y of the commodity contingent on the agents' individual states, lying in the set Y (i.e., a collection of trades, or contracts to offer; contracts of the same type are then pooled and transformed according to the Law of Large Numbers) so as to maximize profits:

$$\max_{y \in Y} \sum_e \sum_s q_s^e y_s^e \qquad (P^f)$$

taking prices q as given.

Definition 3 *A Walrasian equilibrium with symmetric information in the moral hazard economy is given by prices $\bar{q}^e \in \Delta^2$, for all e, a consumption allocation and effort choice $(\bar{c}, \bar{e}) \in \Re_+^2 \times \{h, l\}$, , a production vector $y \in \Re^4$, such that:*

(i) $(\bar{c}, \bar{e})$ solves the agent's optimization problem

$$max_{c,e} \sum_s \pi_s^e u(c_s) - v(e) \qquad (3)$$

s.t.

$$\sum_s \bar{q}_s^e (c_s - w_s) = 0$$

(ii) $\bar{y}$ solves the firms' profit maximization problem (P^f), at prices $\bar{q}$;

(iii) markets clear:

$$(\bar{c}_s^e - w_s) \leq \bar{y}_s^e, \text{ for all } s, e^3 \qquad (4)$$

Definition 4 *A Walrasian equilibrium with symmetric information in the adverse selection economy is given by prices $\bar{q}^e \in \Delta^2$ and a consumption allocation $\bar{c}^e \in \Re_+^2$, for all e, such that:*

(i) $\bar{c}^e$ solves the optimization problem of agents of type e:

$$max_{c^e} \sum_s \pi_s^e u(c_s^e) \qquad (5)$$

s.t.

$$\sum_s \bar{q}_s^e (c_s^e - w_s) = 0$$

(ii) $\bar{y}$ solves the firms' profit maximization problem (P^f), at prices $\bar{q}$;

(iii) markets clear:

$$(\bar{c}_s^e - w_s) \leq \bar{y}_s^e, \text{ for all } s, e \qquad (6)$$

Proposition 1 *Any Walrasian equilibrium with symmetric information in the moral hazard economy is optimal.*

[3] We adopt here the convention that, for $e \neq \bar{e}$, $\bar{c}_s^e = w_s$.

Proposition 2 *Any optimal allocation of a moral hazard economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.*

Proposition 3 *Any Walrasian equilibrium with symmetric information in the adverse selection economy is optimal.*

Proposition 4 *Any optimal allocation of an adverse selection economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.*

4 Incentive Constrained Pareto Optimality

Definition 5 *An allocation* $(\bar{c}, \bar{e}) \in \Re^2_+ \times \{h, l\}$ *of consumption and effort is* incentive constrained optimal *in the moral hazard economy if it solves:*

$$max_{c,e} \sum_s \pi^e_s u(c_s) - v(e) \tag{7}$$

s.t.

$$\sum_s \pi^e_s (c_s - w_s) = 0$$

$$\sum_s \pi^e_s u(c_s) - v(e) \geq \sum_s \pi^{e'}_s u(c_s) - v(e'),\ \forall e, e'$$

Definition 6 *An allocation* $(\bar{c}^h, \bar{c}^l) \in \Re^4_+$ *of consumption is* incentive constrained optimal *in the adverse selection economy if it solves:*

$$max_{c^h, c^l} \sum_e \kappa^e \sum_s \pi^e_s u(c^e_s) \tag{8}$$

s.t.

$$\sum_e \xi^e \sum_s \pi^e_s (c^e_s - w_s) = 0$$

$$\sum_s \pi^h_s u(c^h_s) \geq \sum_s \pi^h_s u(c^l_s)$$

$$\sum_s \pi^l_s u(c^l_s) \geq \sum_s \pi^l_s u(c^h_s)$$

for some $(\varkappa^h, \kappa^l) \in \Re^2_{++}$ *such that* $\kappa^h = 1 - \kappa^l$.

5 Walrasian Equilibria: Fully Observable Trades

We consider first the case in which all the agents' trades are observable, and hence exclusive contracts can be implemented.

For both moral hazard and adverse selection economies, in this case, two different equilibrium concepts have been proposed.

We first introduce Prescott-Townsend equilibria (introduced by Prescott-Townsend (1984) for moral hazard, and extended to encompass adverse selection by Bisin-Gottardi (2000). We then introduce refined non-linear prices equilibria.

5.1 Moral Hazard

We first consider Prescott-Townsend equilibria, as introduced by Prescott-Townsend for moral hazard economies. We then consider refined non-linear prices equilibria, by analogy with the adverse selection concept proposed by Gale (1993) and Dubey-Geanakoplos-Shubik (1995) .

5.1.1 Prescott-Townsend Equilibria

The structure of existing markets is the same as under symmetric information: at the price q_s^e agents and firms can trade claims contingent on the agents' individual state and effort level. To ensure the viability of markets for claims contingent on the agents' effort even though this is now unobservable, the agents' set of admissible trades is suitably restricted to the subset of trades and effort choices which are incentive compatible:

$$Z_{MH} = \left\{ (c,e) \in \Re_+^2 \times \{h,l\} : \sum_s \pi_s^e u\left(c_s\right) - v(e) \geq \sum_s \pi_s^{e'} u\left(c_s\right) - v(e'), \;\; e' \neq e \right\}$$

The incentive constraints require that the agents' choice (c, e) must be such that they prefer (c, e) to any other allocation (c, e'), with $e' \neq e$.

The agents' optimization problem consists, as before, in the choice of a consumption bundle $c \in \Re_+^2$, specifying his level of consumption in the agent's two possible individual states, and an effort level $e \in \{h, l\}$ subject to the budget constraint and, now, to the additional restriction that admissible choices are restricted to lie in the set Z_{MH}:

$$\max_{c,e \in Z_{MH}} \quad \sum_s \pi_s^e u\left(c_s\right) - v(e) \qquad (P_{PT})$$

s.t.

$$\sum_s q_s^e (c_s - w_s) \leq 0$$

Firms' technology Y is the same as in the symmetric information case, and so is the firms' problem:

$$\max_{y \in Y} \sum_e \sum_s q_s^e y_s^e \qquad (P^f)$$

taking prices q as given.

Thus the only difference, with respect to the symmetric information case, generated by the fact that effort is privately observed, is the fact that the consumer's optimization problem is subject to the additional constraint that consumption and effort choices have to be incentive compatible (or that (c,e) are restricted to lie in the set Z_{MH}).

Definition 7 *A PT equilibrium is given by an allocation* $\bar{c} \in \Re_+^2$, *effort level* $\bar{e} \in \{h,l\}$, *a production vector* $\bar{y} \in \Re^4$, *and a price vector* $\bar{q} \in \Delta^3$ *such that:*
(i) $(\bar{c}, \bar{e})$ *solves the agent's optimization problem* (P_{PT}), *at prices* $\bar{q}$
(ii) $\bar{y}$ *solves the firms' profit maximization problem* (P^f), *at prices* $\bar{q}$;
(iii) markets clear:

$$(\bar{c}_s^e - w_s) \leq \bar{y}_s^e, \text{ for all } s,e \qquad (9)$$

5.1.2 Refined Competitive Equilibria with Non-Linear Prices

A fully non-linear system of prices in principle is characterized by the fact that prices are an arbitrary - possibly non-linear - function of the net trades of the consumption good in each state, $c - w = (c_H - w_H, c_L - w_L)$. Each agent solves the following problem:

$$\max_{c_H, c_L, e} \sum_s \pi_s^e u(c_s) - v(e) \qquad (P_{NL,MH})$$

s.t.

$$q\,(c-w) \leq 0$$

No restriction is imposed here on the set of admissible trades and prices do not depend on the unobservable level of effort.

Let $C \equiv [0, \sum_s \pi_s^h w_s]^2$. With no loss of generality we can restrict consumers' possible consumption choices to the compact set C^2. Given the non-convexity of the agents' budget feasible choices of consumption and effort, we shall explicitly allow here for the possibility that they 'randomize' in their choices[4]: for each $c \in C$, then $\lambda(c) \in [0,1]$ be the fraction of agents choosing consumption c; similarly, let $h(c) \in [0,1]$ be the proportion of the agents with consumption level c who choose effort h, for all $c \in C$.

Unlike the previous case, firms are now unable to offer claims directly contingent on the agents' effort level. Thus their problem consists in the choice of

[4]Strictly speaking, the agents' consumption choice problem is non-convex also in PT-equilibria. However there, for the simplicity of the presentation, the definition of equilibrium was presented for the case in which consumers'choices are non-random (and symmetric); also there the price faced by agents choosing high and low effort was different, unlike here.

how much to offer of each contract. The set of all possible contracts is identified by the set of all possible specifications of net payments to the agents in the H and L states (geometrically, all points in the two-dimensional orthant); as for consumers, they can be restricted to the compact set $C - w$. The subset of contracts which is feasible is the subset of contracts which are self-financing, or require a net payment not exceeding zero. In the present situation, where contracts are not directly contingent on the agents' effort level, the net payment depends on the level of effort which firms anticipate will be chosen by consumers trading the contract. For any $y \equiv (y_L, y_H) \in C - w$, let $h^f(y)$ denote the firms' anticipation over the proportion of agents trading this contract choosing effort h.

The set of feasible contracts is then :

$$\mathcal{Y}(h^f(.)) = \{y \in C - w : \sum_s \left(\pi_s^h h^f(y) + \pi_s^l (1 - h^f(y))\right) y_s \leq 0\}$$

while the firms' technology is the set of possible quantities which can be supplied of each feasible contracts:

$$Y^{NL}(h^f(.)) = \left\{\mu : \mathcal{Y}(h^f(.)) \to \Re_+\right\}$$

The technology is still characterized by constant returns to scale and is, now, dependent on the firms' expectation over the agents' effort choice $h^f(.)$.

The firms' problem is then the choice of a vector y, specifying a contract, lying in the set $Y^{NL}(h(.))$ so as to maximize profits:

$$\max_{\mu \in Y^{NL}(h^f(.))} \int_{\mathcal{Y}(h^f(.))} \mu(y) q(y) \qquad (P_{NL}^f)$$

Market clearing requires now that the market for each contract clears, i.e. that for each point in $C - w$ supply by firms equal demand by consumers:

$$\lambda(c) \leq \mu(c - w), \qquad \forall c \in C \qquad (10)$$

In addition, at equilibrium firms' expectations over the agents' effort choices have to be correct::

$$h^f(c - w) = h(c), \qquad \forall c \in C \qquad (11)$$

Note that, when $\lambda(c) = 0$, i.e. when consumers are not choosing the contract yielding consumption c, their effort choice for that level of c is indeterminate: any $h(c) = [0, 1]$ satisfies the agents' optimization problem. In this case, the consistency condition (11) has then no bite; this condition in fact only restricts firms' expectations for the contracts which are traded at equilibrium.

As a consequence, we can have a large variety of equilibria, sustained by different beliefs by firms over the effort levels chosen by agents for the non traded contracts. Some of these equilibria are however supported by beliefs which are clearly 'unreasonable'. To rule them out, and restrict the possible equilibria, a refinement (in the spirit of 'trembling hand') will be introduced.

The refinement can be written naturally. Consider the economy perturbed as follows: for each $c \in C$, a fraction $\epsilon(c) > 0$ of agents, such that

$$\int_C \epsilon(c) = \epsilon,$$

is constrained to choose c (and is then free to pick optimally e). No restriction on the perturbation is imposed other than $\epsilon(c) > 0$ for all $c \in C$; the refinement imposed will thus be very mild.

Indexing the perturbation by ϵ an equilibrium of the perturbed economy is obtained by requiring market clearing as in (11), where now, obviously,

$$\lambda(c) \geq \epsilon(c) > 0, \ \ \forall c \in C.$$

A refined non-linear prices equilibrium is then defined as a competitive equilibrium with non-linear prices which is a limit point of a sequence of equilibria of the perturbed economy, for $\epsilon \to 0$.

Given the constant returns to scale of the firms' technology, the existence of a solution of their choice problem requires $q(y) \leq 0$ for all $y \in \mathcal{Y}(h^f(.))$. Thus, for all y such that $\mu(y) > 0$ we have $q(y) = 0$. Note also that condition (10) imposes a separate market clearing condition for each quantity (contract) traded, i.e. embodies a no cross-subsidization condition across different contracts.

Definition 8 *A NL,MH-equilibrium is given by consumers' and firms' choices* $\lambda(c), h(c)$ *and* $\mu(c)$*, for all* $c \in C$*, and a price map* $q(c)$ *for all* $c \in C$ *such that:*
(i) $\lambda(.), h(.)$ *are a solution of the agent's optimization problem* $(P_{NL,MH})$*, at prices* $q(.)$ *(i.e.,* $\lambda(c) > 0, h(c) > 0 \Rightarrow \sum_s \pi^e_s u(c_s) - v(e) \geq \sum_s \pi^e_s u(c'_s) - v(e')$ *for* $c, c' \in C$*,* $e' \in \{h, l\}$*; analogously if* $\lambda(c) > 0, h(e) = 0$*);* $\int_C \lambda(c) = 1$*;*
(ii) $\mu(.)$ *is a solution of the firms' optimization problem (i.e,* $\mu(y) > 0 \Rightarrow \mu(y) \geq \mu(y')$ *for all* $y, y' \in Y^{NL}(h^f(.))$*, given* $q(.), h^f(.)$*)*

(iii) markets clear, (10);

(iv) firms expectations are consistent with consumers' choices (11).

A NL,MH(ϵ*) is a NL,MH of the perturbed economy in which* $1-\epsilon$ *agents solve problem* $(P_{NL,MH})$*, while* $\epsilon(c)$ *agents solve problem* $(P_{NL,MH})$*, constrained by* $c_L = c$*, for all* $c \in C$*.*

A RNL,MH is a NL,MH such that $(c, e), q(c)$ *are the limit points of the sequence of allocations associated to NL,MH(*ϵ*), for* $\epsilon \to 0$*.*

5.1.3 Strategic Equilibria

We present here the strategic equilibrium notion we shall consider in this paper. Assume the economy is also populated by I financial intermediaries. Each intermediary $i = 1, \dots, I$ can issue J^i contracts (securities), indexed by $j^i = 1, \dots, J^i$; J^i is assumed given, and large.[5] Let $J \equiv \sum_i J^i$; also, we let J, J^i denote the sets of contracts which can be issued as well as their cardinality. A contract is identified by a vector of (possibly negative) payoffs paid by the intermediary to the buyer of the contract, conditionally on the realization of the publicly observable characteristics of the agent trading the contract. More precisely, when effort is unobservable, a contract j is a pair $y^j = (y^{j,H} y^{j,L})$ describing the payoff respectively in state H and L.[6]

Given the set of contracts issued by all intermediaries, agents choose which contracts to enter and which effort level to undertake; their consumption level is then uniquely determined by their choice of contracts. Perfectly anticipating the agents' choices, as a function of the set of contracts available to them, intermediaries strategically choose which contracts to issue (and how many units of each of them), so as to maximize profits.

With regard to the information available to intermediaries over agents' trades, when we are in the case of complete observability of trades, each intermediary is able to perfectly monitor all the transactions an agent makes, and hence to implement exclusive contracts. Each agent can then only choose to buy one of the J contracts available to him. Letting $\{\lambda_{j^i}\}_{j^i \in J^i, i \in I}$ denote the agent's portfolio choices, the problem solved by agents, given the set of contracts $y \equiv (y^{j^i})_{j \in J^i, i \in I}$ issued by intermediaries, can be formally described as follows:

$$\max_{\lambda \in \{0,1\}^J, e \in \{h,l\}, c \in \Re^2_+} \sum_s \pi^e_s u(c_s) - v(e) \text{ s. t.} \qquad (P^S_{NL,MH})$$

$$\lambda_{j^i} = 1 \Rightarrow \lambda_{j'^{i'}} = 0 \ \forall j'^{i'} \neq j^i$$

$$c_s = w_s + \sum_{j^i \in J^i, i \in I} \lambda_{j^i} y^{j^i}_s, \ s \in \{H, L\}$$

Note that the portfolio choices of each agent are restricted to pick only one contract from the set of contracts offered by all intermediaries..

The optimization problem faced by each intermediary $i \in I$, given $(y^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i$, is then[7]:

$$\max_{(y^{j^i})_{j^i \in J^i} \in \Re^{2J^i}} \{- \sum_{j^i \in J^i} [\pi^e_H y^{j^i}_H + \pi^e_L y^{j^i, L}] \lambda_{j^i}\} \text{ s. t.} \qquad (P^{S,f}_{NL,MH})$$

[5] It should be clear from the proof of all the results that the condition that each intermediary can only issue finitely many contracts is never a restriction.

[6] We assume that the trivial contract with zero payoff is always an available choice to the intermediaries. Thus the fact that they are required to issue no less than J^i contracts is clearly not restrictive.

[7] Prices could have also been explicitly introduced in this set-up, as in the earlier models, and allow intermediaries to compete both in the price and the contract they offer. We avoided doing it only for the simplicity of the exposition.

s.t.

$(e, (\lambda_{j^i})_{j^i \in J^i})$ solves problem $(P^S_{NL,MH})$ when $y = [(y^{j^i})_{j^i \in J^i}, (y^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i]$

Thus intermediaries play a simultaneous game, in which the choice variable is the menu of contracts they issue, and perfectly anticipate the agents' choices as a function of the set of contracts issued.[8]

Definition 9 *An equilibrium with strategic intermediaries and complete observability of trades is then an array* $\{(\lambda, e, c), y\}$ *such that:*
(i) (λ, e, c) *solves problem* $(P^S_{NL,MH})$ *given* y,
(ii) $(y^{j^i})_{j^i \in J^i}$ *solves* $(P^{S,f}_{NL,MH})$ *given* $(y^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i$.

5.1.4 Results

Proposition 5 *Any PT equilibrium in the moral hazard economy is incentive constrained optimal.*

Proposition 6 *Any incentive constrained optimal allocation of a moral hazard economy with symmetric information can be decentralized as a PT equilibrium with transfers.*

Proposition 7 *Any PT equilibrium in the moral hazard economy is a RNLMH; and viceversa.*

Proposition 8 *For I sufficiently large, all equilibria with strategic intermediaries and complete observability of trades support the same allocations as PT equilibria.*

5.2 Adverse Selection

We first consider Prescott-Townsend equilibria, as introduced by Bisin-Gottardi (2000) to encompass the adverse selection case. We then consider refined nonlinear prices equilibria, as introduced by Gale (1993) and Dubey-Geanakoplos-Shubik (1995) (though in somewhat different environments).

5.2.1 Prescott-Townsend Equilibria

As in the moral hazard case, the structure of existing amrkets is the same as when information is symmetric. Therefore, we allow prices to depend on the agents' type, even though this is only privately observed: $q^e_s \in \mathbb{R}_+$ is the unit price at which any agent who claims to be of type $e \in \{h, l\}$ can trade

[8] If the consumers' choice, after the intermediaries have decided which contracts to issue, is viewed as a subgame, the strategic equilibrium concept we use is then a subgame perfect Nash equilibrium.

the commodity for delivery in his individual state $s \in \{H, L\}$. To ensure the viability of these markets even though the agents' type is now only privately observable, the set of admissible trades of agents will be restricted by imposing incentive compatibility constraints.

However, the incentive compatibility constraints faced by an agent who claims to be of type i depend on the level of trades by agents of type $e \neq e'$ agents. Thus, unlike with moral hazard, the imposition of incentive compatibility constraints here will impose an externality in the specification of the agents' set of admissible trades. More precisely, let $z^e \equiv \{z_s^e\}_{s\in S}$, $e \in \{h, l\}$, the set of admissible net trades for every agent is then defined as follows:

$$Z(\bar{z}^h, \bar{z}^l) = \left\{ \begin{array}{c} z^h, z^l \in \mathbb{R}^4 : \\ \\ z^h - w \geq 0, z^l - w \geq 0 \\ \\ z^e \neq 0 \Longrightarrow z^{e'} = 0 \qquad \forall e, e' \neq e \in \{h, l\}, \qquad \text{and} \\ \\ \\ \sum_{s\in S} \pi_s^g u\left(w_s + z_s^h\right) \geqslant \sum_{s\in S} \pi_s^g u\left(w_s + \bar{z}_s^l\right) \\ \sum_{s\in S} \pi_s^l u\left(w_s + \bar{z}_s^l\right) \geqslant \sum_{s\in S} \pi_s^l u\left(w_s + z_s^h\right) \end{array} \right\}$$

where $\bar{z}^l, \bar{z}^h$ denote the net trades made in the market by agents (who claim to be) respectively of type l, h, taken as exogenously given.

The above specification reflects the following facts. Every agent can claim to be of type h and trade in the market for the type h; alternatively, he can claim to be of type l and trade in the market for l. If he chooses to trade in the market for h, i.e. $z^h \neq 0$, then he cannot trade in the market for l, $z^l = 0$. Moreover, the level of his net trades in the market for h has to satisfy the incentive compatibility constraints with respect to the level of net trades made by agents who claim to be of type l, $\bar{z}^l$, taken as exogenously given. Such constraints require that agents of type h prefer (at least weakly) z^h to $\bar{z}^l$, and similarly that type l agents prefer $\bar{z}^l$ to z^h. Symmetric restrictions hold if the agent chooses instead to trade in the market for the l types.

The choice problem of an agent of type $e \in \{h, l\}$ has then the following form:

$$\max_{z\in Z(\bar{z}^h, \bar{z}^l)} \quad \sum_{s\in S, e'\in\{h,l\}} \pi_s^e u\left(w_s + z_s^{e'}\right) \qquad (P_{EPT}^e)$$

s.t.

$$\sum_{s\in S, e'\in\{h,l\}} q_s^e z_s^{e'} \leq 0$$

As with moral hazard, the consumers' problem is the same as with symmetric information, except for the restriction imposed on agents' trades. Furthermore, in this case the level of net trades chosen by the other agents (of both types) in the economy enters this problem as it restricts the set of admissible trades, via the incentive compatibility constraints. Thus we have, formally, an externality in the consumption space which is not internalized in the model.

The firms' problem is againthe same as when information is symmetric:

$$\max_{y \in Y} \sum_{e} \sum_{s} q_s^e y_s^e \qquad (P^f)$$

taking prices $q \equiv [q_s^i]_{s\in S}^{i\in\{g,b\}}$ as given.

Following Bisin and Gottardi (2000), we denote the competitive equilibrium here as EPT, for PT equilibrium with an externality.

Definition 10 *An EPT is given by a collection of net trades for each consumers' type* $\{z^l, z^h\}$*, a production vector* y*, a price vector* q *and a pair* $\{\bar{z}^l, \bar{z}^h\}$ *such that:*
(i) $(z^e, 0)$ *solves the optimization problem* (P^e_{EPT}) *of consumers of type* e*, at* $(q, \bar{z}^h, \bar{z}^l)$*, for* $e \in \{h, l\}$*;*
(ii) y *solves the firms' profit maximization problem* (P^f)*, at the prices* q*;*
(iii) markets clear:

$$z_s^e \leq y_s^e, \; e \in \{h, l\}, s \in S \qquad (12)$$

(iv) agents' choices are consistent with the level of trades in the market, taken as given by agents:

$$z_s^e = \bar{z}_s^e, \; e \in \{h, l\}, s \in S$$

Condition (i) requires that, faced with prices q^h and q^l, agents of type h prefer to trade at prices q^h and type l prefers to trade at q^l. This is ensured by the presence of the incentive compatibility constraints in the specification of the agents' set of admissible trades together with the consistency condition (iv).

5.2.2 Refined Competitive Equilibria with Non-Linear Prices

As in the moral hazard case, let $q(c - w)$ be price - possibly non-linear - of the net trades $(c - w)$ of the consumption good in each state. Each agent of type $e \in \{h, l\}$ solves the following problem:

$$\max_{c_H, c_L} \sum_{s} \pi_s^e u(c_s) \qquad (P_{NL,AS})$$

s.t.

$$q\,(c - w) \leq 0$$

With no loss of generality we can restrict again consumers' possible consumption choices to the compact set C, for $C \equiv [0, \sum_s \pi_s^h w_s]^2$.

For each $c \in C$, then $h(c) \in [0, 1]$ $(l(c) \in [0, 1])$ be the fraction of agents of type h (respectively, l) choosing consumption level c.

Firms' problem consists in the choice of how much to offer of each contract, where the set of all possible contracts is the set of all possible specifications of net payments to the agents in the H and L states, which can be restricted to the compact set $C - w$. To determine the subset of contracts which is feasible - or self-financing - we need to specify the proportions of the two types which firms anticipate will choose the contract, as this determine the expected net payment on the contract. For any $y \equiv (y_L, y_H) \in C^2 - w$, let $h^f(y)$ denote the firms' anticipation over the proportion of agents trading this contract who are of type h.

The specification of the set of feasible contracts is then the same as with moral hazard :

$$\mathcal{Y}(h^f(.)) = \{y \in C - w : \sum_s \left(\pi_s^h h^f(y) + \pi_s^l (1 - h^f(y))\right) y_s \leq 0\}$$

and so the firms' technology:

$$Y^{NL}(h^f(.)) = \left\{\mu : \mathcal{Y}(h^f(.)) \to \Re_+\right\}$$

The firms' problem is then the choice of a vector y, specifying a contract, lying in the set $Y^{NL}(h(.))$ so as to maximize profits:

$$\max_{\mu \in Y^{NL}(h^f(.))} \int_{\mathcal{Y}(h^f(.))} \mu(y) q(y) \qquad (P_{NL}^f)$$

Market clearing requires now that the market for each contract clears, i.e. that for each point in $C - w$, supply by firms equal demand by consumers:

$$h(c) + l(c) \leq \mu(c - w), \qquad \forall c \in C \qquad (13)$$

In addition, at equilibrium firms' expectations over the agents' effort choices have to be correct::

$$h^f(c - w) = \begin{cases} \frac{h(c)}{h(c)+l(c)}, \text{ if } h(c) + l(c) > 0 \\ arbitrary, \text{ if } h(c) + l(c) = 0 \end{cases} \qquad \forall c \in C \qquad (14)$$

Note that, when $h(c) + l(c) = 0$, i.e. when no consumer of any type chooses the contract yielding consumption c, the firms' expectation over the proportion of type h agents trading the contract yielding c is indeterminate: any $h^f(c) = [0, 1]$ is consistent with consumers' choices. In this case, the consistency condition (14) has then no bite; this condition in fact only restricts firms' expectations for the contracts which are traded at equilibrium.

As a consequence, we can have a large variety of equilibria, sustained by different beliefs by firms over the proportion of h and l types which would choose any traded contract if this were issued. To restrict the set of possible equilibria, a refinement (in the spirit of 'trembling hand') will be introduced.

As in the case of moral hazard, the perturbed economy is characterized by the fact that a (small) fraction of agents $\epsilon(c) > 0$ is constrained to buy the contract yielding c, for each $c \in C$ such that

$$\int_C \epsilon(c) = \epsilon,$$

However now we also have to specify the composition of this fraction given by h and by l types; different specifications of this composition lead to different equilibrium sets. We will consider here the case where the fraction $\epsilon(c) > 0$ of agents constrained to trade contract c is made entirely by agents of type h, for every c. The equilibria we obtain will clearly depend in this case from this particular specification of the refinement.

Indexing the perturbation by ϵ an equilibrium of the perturbed economy is obtained by requiring market clearing as in (13), where now, obviously,

$$\lambda(c) \geq \epsilon(c) > 0, \ \ \forall c \in C.$$

A refined non-linear prices equilibrium is then defined as a competitive equilibrium with non-linear prices which is a limit point of a sequence of equilibria of the perturbed economy, for $\epsilon \to 0$.

Given the constant returns to scale of the firms' technology, the existence of a solution of their choice problem requires $q(y) \leq 0$ for all $y \in \mathcal{Y}(h^f(.))$. Thus, for all y such that $\mu(y) > 0$ we have $q(y) = 0$. Note also that condition (13) imposes a separate market clearing condition for each quantity (contract) traded, i.e. embodies a no cross-subsidization condition across different contracts.

Definition 11 *A NL,AS-equilibrium is given by choices of the two consumers' types and firms, $l(c), h(c)$ and $\mu(c)$, for all $c \in C$, a price map $q(c)$ and firms' anticipations $h^f(c)$, for all $c \in C$, such that:*
(i) $h(.)$ - resp. $l(.)$ - is a solution of the type h (l) agents optimization problem $(P^h_{NL,AS})$, at prices $q(.)$ (i.e., $h(c) > 0 \Rightarrow \sum_s \ \pi^h_s u(c_s) \geq \sum_s \ \pi^h_s u(c'_s)$ for $c, c' \in C$); analogously for $l(.)$;
(ii) $\mu(.)$ is a solution of the firms' optimization problem (i.e, $\mu(y) > 0 \Rightarrow \mu(y) \geq \mu(y')$ for all $y, y' \in Y^{NL}(h^f(.))$, given $q(.), h^f(.)$)
(iii) markets clear, (13);
(iv) firms expectations are consistent with consumers' choices (14).

A NL,AS(ϵ) is a NL,AS of the perturbed economy in which $1-\epsilon$ agents solve problem $(P_{NL,MH})$, while $\epsilon(c)$ agents solve problem $(P_{NL,MH})$, constrained by $c_L = c$, for all $c \in C$.

A RNL,AS is a NL,AS such that $(c, e), q(c)$ are the limit points of the sequence of allocations associated to NL,AS(ϵ), for $\epsilon \to 0$.

5.2.3 Strategic equilibria

[Maskin Tirole; ...]

5.2.4 Results

Proposition 9 *There exists a unique EPT competitive equilibrium, given by a price vector* $\bar{q}$ *satisfying, for all* e, s :

$$q_s^e = \pi_s^e \tag{15}$$

a production plan $\bar{y}$ *satisfying (**??**), and a consumption allocation* $\{\bar{c}^h, \bar{c}^l\}$ *such that:*
i. $\bar{c}_L^l = \bar{c}_H^l = \pi_L^l w_L + \pi_H^l w_H$;
ii.$(\bar{c}^h - w, 0)$ *solves the optimization problem* (P_{EPT}^h) *of consumers of type h, at* $\bar{q}$, $\bar{z}^h = \bar{c}^h - w, b\bar{z}^l = \bar{c}^l - w$.

Proposition 10 *All EPT equilibrium allocations are efficient within the restricted set of feasible allocations which are incentive compatible and, in addition, satisfy the condition*

$$\sum_s \pi_s^e(c_s^e - w_s) = 0, \text{ for } e \in \{h, l\} \tag{16}$$

On the other hand, a second welfare theorem result holds for the present structure of markets: any incentive efficient consumption allocation can be decentralized as an EPT equilibrium with transfers (possibly dependent on the state but not the agents' type).

Proposition 11 *For any incentive efficient consumption allocation* (c^h, c^l) *there exists a set of transfers* (t_H, t_L) *(common for all types) which are feasible, i.e.,* $\sum_s \left[\xi^h \pi_s^h t_s + \xi^l \pi_s^l t_s\right] \leq 0$, *and such that* (c^h, c^l) *is an EPT equilibrium allocation for the economy under consideration when each agent receives a transfer* (t_H, t_L).

Proposition 12 *Any EPT equilibrium in an adverse selection economy is a RNLAS; and viceversa.*

6 Walrasian Equilibria: Non-Observable Trades

We consider next the case where no information is available over the trades each agent makes, we are thus in an extreme case of lack of exclusivity. In this situation prices cannot vary with the quantity traded by an agent, and quantities cannot be used to try to separate different types/effort choices; competitive equilibria with linear prices are then considered.

6.1 Moral hazard

6.1.1 Competitive equilibria with linear prices

The choice problem of each agent has the following form. The agent chooses a consumption and effort bundle $(c, e) \in \Re_+^2 \times \{h, l\}$, specifying his level of consumption in the agent's two possible individual states, and his effort level subject to the budget constraint:

$$\max_{c,e} \quad \sum_s \pi_s^e u(c_s) - v(e) \qquad (P_{MH}^L)$$

s.t.

$$\sum_s q_s(c_s - w_s) \leq 0$$

The main difference with respect to PT equilibria is that the agent faces now a price which is independent of his effort choice and his trades are not restricted by incentive compatibility. Agents are then free to trade over the whole consumption set at linear prices.

Given the non-convexity of the agents' choice problem we will again allow for the possibility, as in the case of non-linear prices, that at equilibrium, even though all agents are identical, some of them will choose high effort and some low effort. Let c^h and c^l denote then the consumption choice of agents who chose to undertake, respectively, high and low effort, and h (resp. $1-h$) the fraction of the population undertaking high (low) effort[9].

Analogously, firms are unable to issue contingent claims conditional on the agents' effort level or on the level of their trades. There are only two possible contracts available to firms: $\{H, L\}$, i.e. claims contingent on the realization of the H individual state and claims contingent on the realization of L. They can still 'pool' payments made by agents in different states of the world, and by different agents, but are unable to separate them according to the quantity traded by each consumer (and thus, indirectly, of his effort level): firms stand ready to trade any amount consumers wish to trade at the given prices, taking the composition of their trades (i.e. the level of effort undertaken by agents trading the contract) as given and equal to the market composition. More precisely, as in the case of non-linear prices the net payment depends on the level of effort which firms anticipate will be chosen by consumers trading the contract (the difference is that the contract is now not identified by c but simply by H or L; as we said there are onloy two possible contracts). Let h_H^f (resp. h_L^f denote the firms' anticipation over the proportion of agents trading individual contract H (L) choosing effort h.

The firms' technology is then the set of levels of net trades of the two types of contracts by firms which are self-financing (require a net payment not exceeding zero):

$$Y^L(h_H^f, h_L^f) = \{y \in \Re^2 : \sum_s \left(\pi_s^h h_s^f + \pi_s^l(1-h_s^f)\right) y_s \leq 0\}$$

The technology is still characterized by constant returns to scale and dependent on the firms' expectation over the agents' effort choice h_H^f, h_L^f.

[9]Note than now h does not vary as c varies (since trades are non observable, agents cannot separated on the basis of the level of their trades).

The firms' problem is then the choice of a vector y, specifying a contract, lying in the set $Y^L(h_H^f, h_L^f)$ so as to maximize profits:

$$\max_{y \in Y^L(h_H^f, h_L^f)} \sum_s y_s q_s \tag{P_L^f}$$

At equilibrium firms' anticipation over the proportion of agents undertaking high effort for each contract have to be consistent with the market average of agents trading this contract and undertaking high effort:

$$h_H^f = \begin{cases} \frac{h(c_H^h - w_H)}{h(c_H^h - w_H) + (1-h)\left(c_H^l - w_H\right)}, \text{ if } h(c_H^h - w_H) + (1-h)\left(c_H^l - w_H\right) \neq 0 \\ arbitrary, \text{ if } h(c_H^h - w_H) + (1-h)\left(c_H^l - w_H\right) = 0 \end{cases} \tag{17}$$

Definition 12 *A competitive equilibrium with linear prices for the moral hazard economy is given by a price vector $q \in \Delta^1$, consumption allocations $c^h \in \Re_+^2, c^l \in \Re_+^2$ in case of high (resp. low) effort, fractions ζ^h, ζ^l, and a production vector $y \in \Re^2$, such that:*
(i) c^h (resp. c^l) solves the agent's optimization problem (P_{MH}^L), at prices q, when $e = h$ (l); $h > 0$ (resp. < 1) if $\sum_s \pi_s^e u\left(c_s^h\right) - v(h) \geqslant$ (resp. $\leq$) $\sum_s \pi_s^e u\left(c_s^l\right) - v(l)$
(ii) y solves the firms' profit maximization problem (P_L^f), at prices q;
(iii) firms expectations are consistent with consumers' choices, (17).
(iv) markets clear:

$$h(c_s^h - w_s)\pi_s^h + (1-h)(c_s^l - w_s)\pi_s^l \leq y_s, \ s \in S \tag{18}$$

6.1.2 Non-existence

We show here that for an open set of economies in the class described above, competitive equilibria with linear prices do not exist. Helpman-Laffont (1975) already provided an example of an economy with hidden action in which a competitive equilibrium fails to exist. The rationale for the lack of existence in that paper is identified in the lack of convexity of the agents' choice problem. We present here a variant of the Helpman-Laffont non-existence example which allows to see the sources of non existence in a simpler and more transparent form. In particular we will identify the main problem for the existence of competitive equilibria in the fact that, with asymmetric information and linear prices, different claims, corresponding to different levels of the unobservable effort *and* different levels of trade, are aggregated in the market clearing conditions. As we will see in the next section, this is also true in the adverse selection economies where the agents' problem is convex. The role of the non-convexity in moral hazard economies is then primarily to lead to the above heterogeneity of claims in the market clearing conditions.

We can only have three types of competitive equilibria:

1. competitive equilibria with a nonzero level of trade where all agents undertake the same effort level. In this case, as it is immediate to see, equilibrium prices are equal to the probabilities contingent on the effort level undertaken at equilibrium (we will refer then to these as *fair price equilibria):*

   low effort: $q_H = \pi^l_H$, and $\zeta^h = 0$;
   high effort: $q_H = \pi^h_H$, and $\zeta^h = 1$

2. competitive equilibria with a zero level of trade: $c_H = w_H$, $c_L = w_L$ (we will call them *no-trade equilibria*); again they can be distinguished according to the effort level undertaken at equilibrium.

   low effort: $\zeta^h = 0$
   high effort: $\zeta^h = 1$;

3. competitive equilibria where agents are indifferent between high and low effort and a fraction $0 < \zeta^h < 1$ of the population undertakes high effort (we call them *mixed equilibria*). The equilibrium prices are in this case at the level $q_H = q^*$ where agents are indifferent between high and low effort.

For the construction of the non-existence example, we specialize here the preferences of the agents to have the following form:

$$\pi^e_H \ln c^e_H + \pi^e_L \ln c^e_L - v(e)$$

We will characterize then the set of competitive equilibria for any possible combination of parameters $< \pi^h_H,\ \pi^h_H,\ v(e^h),\ v(e^l), w_H, w_L >$ and we will show that for an open set of the parameters no equilibrium exists. A graphical illustration of the argument is provided in Figure ??.

We proceed as follows: we first derive conditions under which equilibria of any of the three type described above exist. We then associate to any point of the parameter space the types of equilibria which exist for this economy. This allows us also to identify and characterize a region of the parameter space to which no competitive equilibrium is associated.

Fair price equilibria with low effort. Solving agent's problem ((P^L_{MH})) when $q_H = \pi^l_H$, we find that if

$$v(e^h) - v(e^l) > \pi^h_H \ \ln\left(\frac{\pi^h_H}{\pi^l_H}\right) + (1-\pi^h_H)\ \ln\left(\frac{1-\pi^h_H}{1-\pi^l_H}\right) \tag{19}$$

the agent will achieve a higher utility level by choosing effort e^l rather than e^H. Therefore, if (19) holds, at the fair price conditionally on the low level of effort all agents will choose the low level of effort (so that $\zeta^h = 0$).

Fair price equilibria with high effort. The existence of such equilibria require that, at the price $q_H = \pi^h_H$, all agents prefer to exert the high level of effort (so that $\zeta^h = 1$); this is indeed the case if

$$v(e^h) - v(e^l) < \pi^l_H \ln\left(\frac{\pi^h_H}{\pi^l_H}\right) + (1-\pi^l_H)\ln\left(\frac{1-\pi^h_H}{1-\pi^l_H}\right) \tag{20}$$

The right hand side of equation has always negative sign. Hence since we assumed that $v(e^h) - v(e^l) > 0$, the above condition can never be satisfied.. Fair price equilibria with high effort never exist.

No-trade equilibria with high effort. Equilibrium prices supporting no trade are

$$\frac{q_H}{1-q_H} = \frac{\pi^h_H w_L}{w_H(1-\pi^h_H)}$$

To have indeed an equilibrium it must be that the utility with a high effort level and no trade is higher than the utility with a low effort level and (possibly) trade at the above prices; i.e.:

$$\begin{aligned}&\pi^h_H \ln w_H + (1-\pi^h_H)\ln w_L - v(e^h) \geq \\ &\geq \pi^l_H \ln(\tfrac{w_H \pi^l_H}{\pi^h_H}) + (1-\pi^l_H)\ln(\tfrac{w_L(1-\pi^l_H)}{1-\pi^h_H}) - v(e^l)\end{aligned}$$

Hence such equilibrium exists if the following inequality is satisfied:

$$\begin{aligned}&v(e^h) - v(e^L) > \\ &> (\pi^h_H - \pi^l_H)(\ln w(1) - \ln w(2)) + \pi^l_H \ln\left(\tfrac{\pi^h_H}{\pi^l_H}\right) + (1-\pi^l_H)\ln\left(\tfrac{1-\pi^h_H}{1-\pi^l_H}\right)\end{aligned}$$

No-trade equilibria with low effort. An argument similar to that of the previous case shows that no-trade equilibria with low effort exist (at prices $\frac{q}{1-q} = \frac{\pi^l_H w(2)}{w(1)(1-\pi^l_H)}$) if the following condition is satisfied:

$$\begin{aligned}&v(e^h) - v(e^L) > \\ &> (\pi^h_H - \pi^l_H)(\ln w(1) - \ln w(2)) + \pi^h_H \ln\left(\tfrac{\pi^h_H}{\pi^l_H}\right) + (1-\pi^h_H)\ln\left(\tfrac{1-\pi^h_H}{1-\pi^l_H}\right)\end{aligned}$$

Mixed equilibria. In order to have $0 < \lambda^h < 1$ agents have to be indifferent between choosing a high and a low level of effort. This condition determines the price q^* for this type of equilibrium, which has to be a solution of the following equation:

$$\begin{aligned}\ln\left(\tfrac{q_H}{1-q_H}\right) = \tfrac{1}{\pi^h_H - \pi^l_H}[\pi^h_H \ \ln\pi^h_H + (1-\pi^h_H)\ \ln(1-\pi^h_H) - \\ -\pi^l_H \ \ln\pi^l_H - (1-\pi^l_H)\ln(1-\pi^l_H) - v(e^h) + v(e^l)]\end{aligned} \tag{21}$$

Several cases are possible in terms of portfolio positions of agents choosing $e = e^l$ or $e = e^h$ in equilibrium. While the enumeration of these cases is rather involved, we can in fact restrict possible equilibria to the following cases.

Lemma 1 *Mixed equilibria satisfy either one of the following:*

- *(*fair on average mixed equilibria*) both low and high effort agents buy insurance; the price is more than fair for the first and less than fair for the second; these equilibria exist if*

$$v(e^H) - v(e^L) - \pi(e^H)\ln\left(\frac{\pi(e^H)}{\pi(e^L)}\right) - (1-\pi(e^H))\ln\left(\frac{1-\pi(e^H)}{1-\pi(e^L)}\right) <$$
$$< \ min\left\{0, (\pi(e^H) - \pi(e^L))\ln\left(\frac{w(1)\pi(e^L)(1-\pi(e^H))}{w(2)\pi(e^H)(1-\pi(e^L))}\right)\right\}$$

- *(*unfair mixed equilibria*) high effort agents sell insurance while low effort buy insurance and the price is less than fair for both; these equilibria exist if*

$$(\pi(e^H) - \pi(e^L))\ln\left(\frac{w(1)}{w(2)}\right) > v(e^H) - v(e^L) - \pi(e^H)\ln\left(\frac{\pi(e^H)}{\pi(e^L)}\right) -$$
$$(1-\pi(e^H))\ln\left(\frac{1-\pi(e^H)}{1-\pi(e^L)}\right) > \ max\left\{0, (\pi(e^H) - \pi(e^L))\ln\left(\frac{w(1)\pi(e^L)(1-\pi(e^H))}{w(2)\pi(e^H)(1-\pi(e^L))}\right)\right\}$$

On the basis of the previous analysis we are now ready to construct a set of conditions under which no equilibria exist, for an open set of parameters $< \pi_H^h,\ \pi_H^h,\ v(e^h),\ v(e^l), w_H, w_L >$.

Proposition 13 *Competitive equilibria do not exist when (all) the following conditions hold:*

i) $\frac{\pi(e^L)(1-\pi(e^H))w(1)}{\pi(e^H)(1-\pi(e^L))w(2)} < 1$

ii) $v(e^H) - v(e^L) - \pi(e^H)\ln\left(\frac{\pi(e^H)}{\pi(e^L)}\right) - (1-\pi(e^H))\ln\left(\frac{1-\pi(e^H)}{1-\pi(e^L)}\right) > (\pi(e^H) - \pi(e^L))\ln\left(\frac{w(1)\pi(e^L)(1-\pi(e^H))}{w(2)\pi(e^H)(1-\pi(e^L))}\right)$

iii) $v(e^H) - v(e^L) - \pi(e^H)\ln\left(\frac{\pi(e^H)}{\pi(e^L)}\right) - (1-\pi(e^H))\ln\left(\frac{1-\pi(e^H)}{1-\pi(e^L)}\right) < 0.$

Moreover, the set of parameters $\left(\pi(e^H), \pi(e^L)v(e^H), v(e^L), w(1), w(2)\right)$ *which satisfy* i-iii) *is an open set.*

Proof of Proposition 13. We only prove the last statement, as the conditions *i-iii)* follow directly from the previous analysis of the different equilibrium types.

Proceed as follows:

- pick $\pi_H^h,\ \pi_H^l$ so that

$$-\pi_H^h \ln\left(\frac{\pi_H^h}{\pi_H^l}\right) - (1-\pi_H^h)\ln\left(\frac{1-\pi_H^h}{1-\pi_H^l}\right) < 0$$

  (such $\pi_H^h,\ \pi_H^l$ always exist, since the expression on the lhs has a maximum value of 0 for $\pi_H^h = \pi_H^l$)

- then $v(e^h) - v(e^l)$ can be used to control conditions (ii) and (iii) (given that the rhs of the second is < 0, which is implied by conditon (i))

- finally w_H and w_L can be used to control condition (i), given that $\frac{\pi_H^l(1-\pi_H^h)}{\pi_H^h(1-\pi_H^l)} < 1$. $\diamondsuit$

Conditions *i-iii)* have a clear 'economic' interpretation. First, notice that:

- agents demand for insurance (i.e. the choice of $c_L - w_L$) is monotonically increasing in $q_H/(1-q_H)$ both for agents exerting high and low effort;

- agents' preference for low versus high effort is also monotonically increasing in $q_H/(1-q_H)$, i.e. at prices $q > q^*$ agents prefer e^l while at price $q < q^*$ agents prefer e^h. We have in fact $U^L - U^H = k + [\pi_H^h - \pi_H^l]\ln(q_H/(1-q_H))$, where k is a term independent of q_H.

Condition *i)* says that at prices which are fair conditional on a low effort level agents exerting high effort choose to sell insurance. Condition *ii)* says that at the price q^* at which agents are indifferent between high and low effort agents exerting the high level of effort choose to sell insurance, or - given the above - that $q^* < \frac{w_L\pi_H^h}{w_H(1-\pi_H^h)}$, i.e. q^* is less than the price at which agents exerting high level effort choose a zero level of trade. The first property implies that there does not exist a fair on average mixed equilibrium. The second property implies, given what we said above, that at the price $\frac{w_L\pi_H^h}{w_H(1-\pi_H^h)}$ agents will prefer a low level of effort so that a no trade equilibrium with high effort level does not exist.

Condition *iii)* then says that $q^* > \pi_H^l$, so that at prices $\pi_H^l/(1-\pi_H^l)$ agents prefer high level of effort, and a fair price equilibrium does not exist. The same will be true for the price supporting no trade conditional on e^l, since this price will be lower than $\pi_H^l/(1-\pi_H^l)$, and hence agents will prefer a high level of effort. Moreover, no unfair mixed strategy equilibrium exists, since this requires $q^* < \pi_H^l$. The above also suggest that an alternative classification of the regions where different equilibria exist can be given in terms of the value of q^*. Also, that if we limit our attention to fair on average mixed equilibria and pure fair equilibria, these never coexist (and we will see later that the other equilibria can be eliminated by refinements of the trembling hand type, and/or they are Pareto dominated).

6.1.3 Competitive equilibria with bid-ask spreads

We show next that if prices allow for a bid-ask spread but are otherwise linear competitive equilibria always exist. A bid-ask spread introduces a - indeed minimal - form on non-linearity in the price schedule, which allows to have a different price for buyers and sellers. The informational requirements for it are quite minimal, as it suffice to be able to observe whether in each transaction the agent makes, he buys or sells.

Let q_H^+ and q_H^- denote, respectively, the buying and selling price of the security paying off if state H occurs; q_L^+ and q_L^- denote the buying and selling prices of the security paying off if state L occurs. Let $(x)^+$ denote $\max(0,x)$ and $(x)^-$ denote $\min(0,x)$.In the presence of bid-ask spreads, the agents' choice

problem becomes:

$$\max_{c,e} \quad \sum_s \pi_s^e u(c_s) - v(e) \qquad (P_{MH}^{BAS})$$

s.t.

$$\sum_s \left[q_s^+ (c_s - w_s)^+ + q_s^- (c_s - w_s)^- \right] \leq 0$$

Similarly, the firms ('pooling') technology has to be modified to reflect the change in the structure of available individual contracts:

$$Y = \{ y \in \Re^4 : \begin{array}{l} \sum_s [y_s^+ - (\zeta^h (c_s^h - w_s)^+ \pi_s^h + \xi^l (c_s^l - w_s)^+ \pi_s^l)] \leq 0 \\ \sum_s [y_s^- - (\zeta^h (c_s^h - w_s)^- \pi_s^h + \xi^l (c_s^l - w_s)^- \pi_s^l)] \geqslant 0 \end{array} \}$$

and their problem is then the choice of a vector y in the set Y such as to maximize profits:

$$\sum_s (q_s^+ y_s^+ + q_s^- y_s^-) \qquad (P_{MH}^{f,BAS})$$

Definition 13 *A competitive equilibrium with bid-ask spreads is given by a price vector $q \in \Delta^3$, consumption allocations $c^h \in \Re_+^2, c^l \in \Re_+^2$ in case of high (resp. low) effort, fractions ζ^h, ζ^l, and a production vector $y \in \Re^4$, such that:*
(i) c^h (resp. c^l) solves the agent's optimization problem (P_{MH}^{BAS}), at prices q, when e is restricted to equal h (l)

(ii) $\zeta^h > 0$ (resp. < 1) if $\sum_s \pi_s^e u\left(c_s^h\right) - v(h) \geqslant$ (resp. $\leq$) $\sum_s \pi_s^e u\left(c_s^l\right) - v(l)$
(ii) y solves the firms' profit maximization problem $(P_{MH}^{f,BAS})$, at prices q;
(iii) markets clear:

$$\zeta^h \left[(c_s^h - w_s)^+ - (c_s^h - w_s)^- \right] \pi_s^h + \zeta^l \left[(c_s^l - w_s)^+ - (c_s^l - w_s)^- \right] \pi_s^l \leq y_s^+ - y_s^-, \; s \in S \tag{22}$$

Evidently, all competitive equilibria with linear prices are also equilibria with bid-ask sprads for a zero spread. On the other hand, there will be other equilibria with a non-zero spread. In particular, it is easy to see that there will always be equilibria where no trade holds, supported by a sufficiently high spread. As in the case of competitive equilibria with non-linear prices it is then of interest to restrict our attention to 'refined' equilibria.

Restrict $c_L \in C \equiv [0, \sum_s \pi_s^h w_s]$. Consider the economy perturbed as follows: a fraction $\epsilon^+ > 0$ of agents is constrained to choose a level of consumption such that $c_L \geqslant \omega_L$ (i.e. cannot sell insurance, and is then free to pick optimally c and e, subject to the budget constraint). Similarly, a fraction $\epsilon^- > 0$ of agents is constrained to choose a level of consumption such that $c_L \leq \omega_L$ (i.e. cannot buy insurance).

Indexing the perturbation by $\epsilon = \epsilon^+ + \epsilon^-$, an equilibrium of the perturbed economy is obtained by requiring market clearing as in (22).

A refined equilibrium with bid-ask spreads is then defined as a competitive equilibrium with bid-ask spreads which is a limit point of a sequence of equilibria of the perturbed economy, for $\epsilon \to 0$.

It is fairly easy to see that competitive equilibria with bid-ask spreads, if they exist, can only be of the following three types:

1. equilibria with low effort and full insurance (i.e. $c_H = c_L = \pi^l_H w_H + (1 - \pi^l_H) w_L, e = e^l$), purchased at the fair price $q^-_H = \pi^l_H, q^+_L = 1 - \pi^l_H$;
2. equilibria with no trade and high effort (i.e. $c_H = w_H, c_L = w_L, e = e^h$);
3. 'mixed' equilibria, where a fraction of the agents in the population exert high effort, while the others exert low effort, and both buy insurance, at the same price (i.e. $c^h_H < w_H, c^l_H < w_H, \frac{w_H - c^l_H}{c^l_L - w_L} = \frac{w_H - c^h_H}{c^h_L - w_L}$), though in different amounts.

Proposition 14 *A refined competitive equilibrium with bid-ask spreads always exists.*

6.1.4 Strategic equilibria

We consider here the strategic equilibrium notion for the case in which intermediaries have no information available over agents' trades:*:* each intermediary cannot observe any of the trades an agent makes with other intermediaries, nor whether agents engage in re-trading fractions or multiples of the contracts they purchased (or sold) from them. Thus intermediaries can only separate the buying and selling positions of each agent in their own contracts.

Let $\{\lambda_{j^i}\}_{j^i \in J^i, i \in I}$ denote the agent's portfolio choices, the problem solved by agents, given the set of contracts $d \equiv (d^{j^i})_{j \in J^i, i \in I}$ issued by intermediaries, can be formally described as follows:

$$\max_{\lambda \in \Re^J_+, e \in \{h,l\}, c \in \Re^2_+} \sum_s \pi^e_s u(c_s) - v(e) \text{ s. t.} \qquad (P^S_{LMH})$$

$$c_s = w_s + \sum_{j^i \in J^i, i \in I} \lambda_{j^i} d^{j^i, s}, \; s \in \{H, L\}$$

Note that the portfolio choices of each agent are restricted to be non-negative. This is without loss of generality because the intermediaries can distinguish the buying and selling positions of each agent, and selling positions can also be described as buying positions of contracts with negative payoffs.

The optimization problem faced by each intermediary $i \in I$, given $(d^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i$, is then:

$$\max_{(d^{j^i})_{j^i \in J^i} \in \Re^{2J^i}} \{- \sum_{j^i \in J^i} [\pi_H^e d^{j^i,H} + (1-\pi_H^e) d^{j^i,L}] \lambda_{j^i}\} \text{ s. t.} \tag{23}$$

$(e, (\lambda_{j^i})_{j^i \in J^i})$ solves problem (P_{LMH}^S) when $d = [(d^{j^i})_{j^i \in J^i}, (d^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i]$

Thus intermediaries play a simultaneous game, in which the choice variable is the menu of contracts they issue, and perfectly anticipate the agents' choices as a function of the set of contracts issued.[10]

Definition 14 *An equilibrium with strategic intermediaries and no observability of trades is then an array* $\{(\lambda, e, c), d\}$ *such that:*
(i) (λ, e, c) *solves problem* (P_{LMH}^S) *given* d,
(ii) $(d^{j^i})_{j^i \in J^i}$ *solves (23) given* $(d^{j^{i'}})_{j^{i'} \in J^{i'}}, i' \neq i$.

Proposition 15 *For* I *sufficiently large, all strategic equilibria with no observability of trades are characterized by the same allocation as the refined competitive equilibria with bid-ask spread.*

Proof.

6.1.5 Efficiency

Competitive equilibria with bid-ask spreads in moral hazard economies have rather poor efficiency properties. Both Pareto efficient and incentive efficient allocations are in fact characterized (at least for the open set of parameter values for which asymmetric information is not irrelevant, i.e. for which Pareto efficient allocations are not incentive efficient) by a high effort level and non-zero level of trades. On the other hand, competitive equilibria with linear prices, as we saw, only support a high effort level with a zero level of trade (hence at the cost of not providing the agents with any insurance).

Thus competitive equilibria with linear prices are typically incentive inefficient. This should not come as a surprise though: as we already saw the decentralization of incentive efficient allocations requires the observability of trades. It is then informationally more demanding than the equilibrium notion we are considering here which was based on the fact that agents' trades are not observable.

[10] If the consumers' choice, after the intermediaries have decided which contracts to issue, is viewed as a subgame, the strategic equilibrium concept we use is then a subgame perfect Nash equilibrium.

It is natural then to ask if the competitive equilibria with linear prices (or bid-ask spreads) studied in this section satisfy a weaker form of efficiency, where the implementation of attainable reallocations does not require the use of more information over agents' trades than the information implicit in the equilibrium being considered (none, as we argued if contracts can be traded at linear prices). We will impose then a further restriction on feasible allocations - in addition to resource feasibility and incentive compatibility - to reflect the non-observability of trades.

Two different formulations of this additional constraint, and hence two different notions of third best or constrained efficiency will be proposed and examined here.

Let $U(e^h, c^h) = \sum_s \pi_s^h u(c_s^h)$ and $U(e^l, c^l) = \sum_s \pi_s^l u(c_s^l)$.

Definition 15 *A* (A) - third best efficient allocation *is the tuple* $< c^H, e^H, c^L, e^L, \lambda^H, beta(h) >$ *which maximizes:*

$$\beta^H U(e^H, c^H) + (1 - \beta^H) U(e^L, c^L) \tag{24}$$

subject to:

feasibility:

$$\begin{array}{l} \lambda^H \pi(e^H)(c^H(1) - w(1)) + (1 - \lambda^H)\pi(e^L)(c^L(1) - w(1)) + \\ +\lambda^H(1 - \pi(e^H))(c^H(2) - w(2)) + (1 - \lambda^H)(1 - \pi(e^L))(c^L(2) - w(2)) = 0 \\ c^H(2) \geq w(2) \ \textit{and} \ c^L(2) \geq w(2) \end{array}$$

incentive compatibility and non-observability of trades:

$U(e^h, c^h) \geqslant U(e^{'}, \hat{c}^h)$, for any $\hat{c}^h = w^H + \lambda^h \begin{pmatrix} c_1^H - w(1) \\ c_2^H - w(2) \end{pmatrix} + \lambda^l \begin{pmatrix} c_1^L - w(1) \\ c_2^L - w(2) \end{pmatrix}$,

for all $\lambda^h, \lambda^l \geqslant 0, e' \in \{h, l\}$

$U(e^l, c^l) \geqslant U(e^{'}, \hat{c}^l)$, for any $\hat{c}^l = w^H + \lambda^h \begin{pmatrix} c_1^H - w(1) \\ c_2^H - w(2) \end{pmatrix} + \lambda^l \begin{pmatrix} c_1^L - w(1) \\ c_2^L - w(2) \end{pmatrix}$,

for all $\lambda^h, \lambda^l \geqslant 0, e' \in \{h, l\}$.

It is clear that the set of feasible allocations according to the above criterion, i.e. of the allocations that satisfy the above conditions, is only a slight generalization of the set of competitive equilibrium allocations of an economy. At any feasible allocation, the level of net trades of agents undertaking high and low effort implicitly defines in fact relative prices $\begin{pmatrix} c_1^H - w(1) \\ c_2^H - w(2) \end{pmatrix}, \begin{pmatrix} c_1^L - w(1) \\ c_2^L - w(2) \end{pmatrix}$; the candidate allocation has to be the optimal choice of the agents when they are free to pick their optimal level of trades subject to the only constraint of preserving the sign of their trades. The only additional instrument given to the planner relative to the auctioneer is in fact to close down markets corresponding to directions of trades (purchases or sales of a contract) which are not used. Note, however, that at non-refined competitive equilibria it is always possible to preclude a direction of trade by selecting a price which is sufficiently unfavorable price to agents to preclude their trading.

As a consequence, all competitive equilibrium allocations which are Pareto ranked (i.e. are Pareto inferior to other equilibrium allocations) are (A) - constrained inefficient. On the other hand, all competitive equilibria which are not Pareto ranked are (A) - constrained efficient. Therefore:

Proposition 16 *There always exist competitive equilibria with bid-ask spreads which are (A) constrained efficient.*

We will next consider another notion of constrained efficiency where a larger set of attainable reallocations is considered, still consistent with the requirement that no information over agents' trades can be used. In particular we consider now as feasible any allocations which can be attained as a competitive equilibrium with linear prices and proportional and lump-sum taxes and transfers. Thus the feasible set is never a singleton and an admissible reallocation can typically induce a change in equilibrium prices; this is the key factor in generating an improvement. The intuition for this result lies in the fact that an externality is generated in the model we consider, since the payoff to the uninformed party of a transaction (say buyers of a contract) is endogenously determined at equilibrium by the aggregate of the decisions of sellers (as earlier noted by Greenwald and Stiglitz (1986)). Hence it is possible to have an improvement by inducing a change in these decisions which will improve the 'quality' of this payoff.

In particular we will consider the set of possible reallocations of resources which are generated by introducing:

- proportional taxes (or transfers) t_s on the purchase of contract s, $s \in \{H, L\}$,
- a lump-sum tax (or transfer) T, uniform across all agents,

such that $T = \sum_s \left[(c_s^h - \omega_s)\xi^h + (c_s^l - \omega_s)(1-\xi^h)\right] t_s$ (budget balance), and looking then at the associated competitive equilibrium.

Clearly, the implementation of any such reallocation requires no information over agents' trades or types.

Definition 16 *A competitive equilibrium is said to be* (B) - constrained Pareto efficient *if there exist no collection of proportional and lump sum taxes/transfers* $((t_s)_{s\in\{H,L\}}, T)$ *such that the associated competitive equilibrium allocation constitutes a Pareto improvement.*

Proposition 17 *Competitive equilibria with linear prices (or bid-ask spreads) are, typically, (B) - constrained inefficient.*

[(the general idea is that the composition of the 'pool' of agents (as characterized by their effort level) who trade contract s can be modified, by perturbing the economy, e.g. with taxes and subsidies (the vector of the average deliveries made by sellers moves, locally, in a space of at least two dimensions). This can happen because agents making different effort levels change their level of trade of the contract or because they change their effort level.

In particular, for each of the three types of equilibria which can exist with bid-ask spreads we can show how an improvement can be constructed. For mixed equilibria, we should be able to find a local improvement by taxing or subsidizing insurance according to the relative price elasticities of the demand for insurance of agents undertaking high vs. low effort. For pure equilibria with no trade and high effort or low effort and full insurance, where the equilibrium price for buying insurance is the low price corresponding to low effort, an improvement (non local in this case) can be constructed by trying to induce agents to buy insurance and still exercise the high effort, and for this we may have to tax insurance purchases.

6.2 Adverse Selection

6.2.1 Competitive equilibria with linear prices

The choice problem of each type of agent has the following form: agent of type $e \in \{H, L\}$ chooses a consumption bundle $c^e \in \Re^2_+$ subject to the budget constraint and the incentive compatibility constraint:

$$\max_{c,e} \sum_s \pi_s^e u(c_s) \qquad (P_{AS}^{LP,e})$$

s.t.

$$\sum_s q_s(c_s^e - w_s) \leq 0$$

The main difference with respect to EPT equilibria is that the agent faces a price which is independent of his type and his trades are not restricted by incentive compatibility. Agents are then free to trade over the whole consumption set at linear prices.

Firms are now no longer able to issue contingent claims conditional on the agents' type. They can still 'pool' payments made by agents in different states of the world, but are unable to separate them according to the type of the agents. Thus their main choice is how many units of each of the two existing contingent claims to trade, taking into account the fact that the average payoff will depend on the level of trade chosen by agents of each type, i.e. they operate with. the following constant returns to scale technology:

$$Y = \{y \in \Re^2 : \sum_s [y_s - (\xi^h (c_s^h - w_s)\pi_s^h + \xi^l (c_s^l - w_s)\pi_s^l)] \leq 0\}$$

where $y \equiv [y_s]_{s \in S}$.

The firms' problem is then the choice of a vector y of the commodity contingent on the agents' individual states, lying in the set Y (i.e., which can be generated by pooling contracts of the same type and transforming them according to the Law of Large Numbers) so as to maximize profits:

$$\sum_s q_s y_s \qquad (P_{AS}^{f,LP})$$

taking prices q as given.

Definition 17 *A competitive equilibrium with linear prices is given by a price vector $q \in \Delta^1$, consumption allocations $c^h \in \Re^2_+, c^l \in \Re^2_+$ for the high (resp. low) type agents, and a production vector $y \in \Re^2$, such that:*
(i) c^e solves the agent's optimization problem $(P_{AS}^{LP,e})$, at prices q, for e $\{h, l\}$,
(ii) y solves the firms' profit maximization problem $(P_{AS}^{f,LP})$, at prices q;
(iii) markets clear:

$$\xi^h(c_s^h - w_s)\pi_s^h + \xi^l(c_s^l - w_s)\pi_s^l \leq y_s, \; s \in S \qquad (25)$$

6.2.2 Non existence

The agents' choice problem in the adverse selection economy is clearly convex. We will show here that existence of competitive equilibria may fail even in this case, and non-existence is a robust phenomenon.

For the construction of the non-existence example, we specialize here again the preferences of the agents to have the following form:

$$\pi_H^e \ln c_H^e + \pi_L^e \ln c_L^e$$

In this case the agents' utility maximization problem can be easily solved and yields an explicit expression of the demand for consumption in the two idiosyncratic states (respectively for agents receiving signal g and b):

$$\begin{array}{l} c_H^h = \pi_g \left(\frac{qw(1)+(1-q)w(2)}{q} \right) \\ c_L^h = (1-\pi_g) \left(\frac{qw(1)+(1-q)w(2)}{1-q} \right) \\ c(1;b) = \pi_b \left(\frac{qw(1)+(1-q)w(2)}{q} \right) \\ c(2;b) = (1-\pi_b) \left(\frac{qw(1)+(1-q)w(2)}{1-q} \right) \end{array} \qquad (26)$$

The market-clearing condition (or equivalently the overall zero-profit condition for the two contracts) is:

$$\begin{aligned} & c(1;g)\pi_g + c(1;b)\pi_b + (1-\pi_g)c(2;g) + (1-\pi_b)c(2;b) \qquad (27) \\ = \; & w(1)\pi_g + w(1)\pi_b + (1-\pi_g)w(2) + (1-\pi_b)w(2) \end{aligned}$$

For this economy the set of no-arbitrage prices is non-empty, and is given by all prices $q \in (0,1)$. We will show that, nonetheless, for an open set of parameters, a competitive equilibrium for the economy described above does not exist.

The excess demand function (equivalently the overall profit function) we obtain from (26) is continuous, for all $q \in (0,1)$. However, when $\frac{w(2)\pi_g}{w(1)(1-\pi_g)} > \frac{\pi_b}{(1-\pi_b)}$ this function has a negative value both when $\frac{q}{1-q} > \frac{\pi_g}{(1-\pi_g)}$ and when $\frac{q}{1-q} < \frac{w(2)\pi_b}{w(1)(1-\pi_b)}$; it is easy to see in fact, from the expressions of the agents' demand, that in the first case agents will be buying insurance, no matter what is the signal received, and will do this at more than fair terms, while the reverse happens in the second case, so that profits will be negative in both situations. For intermediate values of the relative price ($\frac{w(2)\pi_b}{w(1)(1-\pi_b)} < \frac{q}{1-q} < \frac{\pi_g}{(1-\pi_g)}$) the sign of aggregate demand cannot be unambiguously determined without further information on the parameter values of the economy. This already suggests that, for some parameter values, aggregate excess demand (profits) may be negative for all prices. Notice that this fact is a clear manifestation of the feasibility problem we discussed in the previous section.

We will now show that we can find an open set of parameter values (in the region $\frac{w(2)\pi_g}{w(1)(1-\pi_g)} > \frac{\pi_b}{(1-\pi_b)}$) for which the above indeed happens and no equilibrium exists.

Let $w(1) = 0.8, w(2) = 0.2, \pi_b = .2$, and $\pi_g = 0.2 + \epsilon$, $\epsilon > 0$. Solving the equations (26) and (27) for equilibrium prices and allocations we find:

$$
\begin{aligned}
&c(1;g) = -.04\frac{-550\epsilon^2-48-355\epsilon+125\epsilon^3+480\epsilon\rho+128\rho+500\rho\epsilon^3-700\rho\epsilon^2}{2+10\epsilon+25\epsilon^2} \\
&c(2;g) = .64\rho + .16 - .8\epsilon\rho - .2\epsilon \\
&c(1;b) = -.04\frac{-115\epsilon+25\epsilon^2-48+128\rho-160\epsilon\rho+100\rho\epsilon^2}{2+10\epsilon+25\epsilon^2} \\
&c(2;b) = .64\rho + .16 \\
&\tfrac{q}{1-q} = \rho
\end{aligned}
$$

where ρ takes one of the two following values:

$$\frac{\left(40+75\epsilon-125\epsilon^2+\sqrt{(576+2160\epsilon-11575\epsilon^2-6750\epsilon^3+5625\epsilon^4)}\right)}{2\left(128-160\epsilon+100\epsilon^2\right)}$$
$$\frac{\left(40+75\epsilon-125\epsilon^2-\sqrt{(576+2160\epsilon-11575\epsilon^2-6750\epsilon^3+5625\epsilon^4)}\right)}{2\left(128-160\epsilon+100\epsilon^2\right)}$$

Straightforward computations reveal that, when $\epsilon > 0.3$, no real-valued solution exist for equilibrium prices and allocations (as the argument of the square root appearing in the above expression determining ρ, $(576 + 2160\epsilon - 11575\epsilon^2 - 6750\epsilon^3 + 5625\epsilon^4)$), has in that case a negative value).

Notice that, with the above parameter values the condition $\frac{w(2)\pi_g}{w(1)(1-\pi_g)} > \frac{\pi_b}{(1-\pi_b)}$ reduces to $\epsilon > 0.3$; hence an equilibrium never exists in this region. It is then immediate to see that perturbing the values of the parameters does not allow to restore existence, so equilibria fail to exist for an open set of parameter values.

On the other hand, when $\frac{w(2)\pi_g}{w(1)(1-\pi_g)} < \frac{\pi_b}{(1-\pi_b)}$ we find that at the prices $\frac{q}{1-q} = \frac{\pi_b}{(1-\pi_b)}$ overall profits are positive (agents receiving signal b buy insurance at fair terms, while agents with signal g also buy insurance but at less than fair terms). By the continuity of the profit function we conclude that an equilibrium always exist in this region.

In particular, for the above parameter specification as we already saw two admissible equilibrium solutions exist when $\epsilon < 0.3$. Moreover, it can be easily seen that the two competitive equilibria we obtain in this region are always Pareto ranked.

To better understand the properties of the competitive equilibria we obtain and more generally of the equilibrium structure of the economy, consider the solutions we get when $\epsilon = 0$ (in this case the signal received by the agents is totally uninformative, information is then symmetric):

$$\begin{aligned}(i)\ c(1;g) &= c(2;g) = c(1;b) = c(2;b) = .32;\ \frac{q}{1-q} = .25 \\ (ii)\ c(1;g) &= .8; c(2;g) = .2; c(1;b) = .8; c(2;b) = .2;\ \frac{q}{1-q} = .0625\end{aligned}$$

The equilibrium in (i) is characterized by the presence of full insurance at fair prices (and is, evidently, Pareto efficient), while equilibrium (ii) has a zero level of trades for all agents.

The system of equilibrium equations (26) and (27) is clearly regular at every solution. Hence the two competitive equilibria we found with adverse selection, i.e. in the region $0.3 > \epsilon > 0$, arise by continuity (as ϵ is varied away from 0) from the two solutions obtained for the economy with symmetric information, the Pareto efficient and the no trade equilibrium.

6.2.3 Competitive equilibria with bid-ask spreads

We show next that if prices allow for a bid-ask spread but are otherwise linear competitive equilibria always exist. A bid-ask spread introduces a - indeed minimal - form on non-linearity in the price schedule, which allows to have a different price for buyers and sellers. The informational requirements for it are quite minimal, as it suffice to be able to observe whether in each transaction the agent makes, he buys or sells.

Let q_H^+ and q_H^- denote, respectively, the buying and selling price of the security paying off if state H occurs; q_L^+ and q_L^- denote the buying and selling prices of the security paying off if state L occurs. Let $(x)^+$ denote $\max(0, x)$ and $(x)^-$ denote $\min(0, x)$.In the presence of bid-ask spreads, the choice problem of agents of type e becomes:

$$\max_{c,e} \sum_s \pi_s^e u(c_s) \qquad (P_{AS}^{BAS,e})$$

s.t.

$$\sum_s \left[q_s^+(c_s^e - w_s)^+ + q_s^-(c_s^e - w_s)^-\right] \leq 0$$

Similarly, the firms ('pooling') technology has to be midified to reflect the change in the structure of available individual contracts:

$$Y = \{y \in \Re^4 : \begin{array}{l} \sum_s [y_s^+ - (\xi^h(c_s^h - w_s)^+\pi_s^h + \xi^l(c_s^l - w_s)^+\pi_s^l)] \leq 0 \\ \sum_s [y_s^- - (\xi^h(c_s^h - w_s)^-\pi_s^h + \xi^l(c_s^l - w_s)^-\pi_s^l)] \geqslant 0 \end{array} \}$$

and their problem is then the choice of a vector y in the set Y such as to maximize profits:

$$\sum_s (q_s^+ y_s^+ + q_s^- y_s^-) \qquad (P_{MH}^{f,BAS})$$

Definition 18 *A competitive equilibrium with bid-ask spreads is given by a price vector $q \in \Delta^3$, consumption allocations $c^h \in \Re_+^2, c^l \in \Re_+^2$, and a production vector $y \in \Re^4$, such that:*
(i) c^e solves the agent's optimization problem ($P_{AS}^{BAS,e}$), at prices q, for $e \in \{h.l\}$
(ii) y solves the firms' profit maximization problem ($P_{AS}^{f,BAS}$), at prices q;
(iii) markets clear:

$$\xi^h\left[(c_s^h - w_s)^+ - (c_s^h - w_s)^-\right]\pi_s^h + \xi^l\left[(c_s^l - w_s)^+ - (c_s^l - w_s)^-\right]\pi_s^l \leq y_s^+ - y_s^-,\ s \in S \qquad (28)$$

Evidently, all competitive equilibria with linear prices are also equilibria with bid-ask sprads for a zero spread. On the other hand, there will be other equilibria with a non-zero spread. In particular, it is easy to see that there will always be equilibria where no trade holds, supported by a sufficiently high spread. As in the case of competitive equilibria with non-linear prices it is then of interest to restrict our attention to 'refined' equilibria.

Restrict $c_L \in C \equiv [0, \sum_s \pi_s^h w_s]$. Consider the economy perturbed as follows: a fraction $\epsilon^+ > 0$ of agents of type h is constrained to choose a level of consumption such that $c_L \geqslant \omega_L$ (i.e. cannot sell insurance, and is then free to pick optimally c and e, subject to the budget constraint). Similarly, a fraction $\epsilon^- > 0$ of agents also of type h is constrained to choose a level of consumption such that $c_L \leq \omega_L$ (i.e. cannot buy insurance).

Indexing the perturbation by $\epsilon = \epsilon^+ + \epsilon^-$, an equilibrium of the perturbed economy is obtained by requiring market clearing as in (22).

A refined equilibrium with bid-ask spreads is then defined as a competitive equilibrium with bid-ask spreads which is a limit point of a sequence of equilibria of the perturbed economy, for $\epsilon \to 0$.

It is fairly easy to see that competitive equilibria with bid-ask spreads, if they exist, can only be of the following two types:

1. equilibria where only the agents of low type trade a nonzero amount and prices are fair wrt to low type (i.e. $c_H^l = c_L^l = \pi_H^l w_H + (1 - \pi_H^l) w_L$), $q_H^- = \pi_H^l, q_L^+ = 1 - \pi_H^l$);
2. 'mixed' equilibria, where both the high and the low types buy insurance, at the same price (i.e. $c_H^h < w_H, c_H^l < w_H, \frac{w_H - c_H^l}{c_L^l - w_L} = \frac{w_H - c_H^h}{c_L^h - w_L}$), though in different amounts.

Proposition 18 *A refined competitive equilibrium with bid-ask spreads always exists.*

6.2.4 Strategic equilibria

[...]

6.2.5 Efficiency

Competitive equilibria with bid-ask spreads also in adverse selection economies have rather poor efficiency properties. Incentive efficient allocations are in fact characterized by full insurance for the agents of one type and partial insurance or overinsurance for the agents of the other type. On the other hand, as we saw, at a competitive equilibrium with linear prices we can only have full insurance for one (the low) type when the other (high) type have a zero level of trade.

Thus competitive equilibria with linear prices are typically incentive inefficient. As in the case of moral hazard we can ask then whether competitive equilibria with linear prices (or bid-ask spreads) satisfy a weaker form of efficiency, where the implementation of attainable reallocations requires no information over the agents' trades, as in the equilibrium being considered. We will consider here for simplicity only the second of the notions of constrained efficiency examined before, where the set of feasible allocations are the allocations which can be attained as competitive equilibria with:

- proportional taxes (or transfers) t_s on the purchase of contract s, $s \in \{H, L\}$,
- a lump-sum tax (or transfer) T, uniform across all agents,

such that $T = \sum_s \left[(c_s^h - \omega_s)\xi^h + (c_s^l - \omega_s)(1 - \xi^h)\right] t_s$ (budget balance).

Clearly, the implementation of any such reallocation requires no information over agents' trades or types.

Definition 19 *A competitive equilibrium is said to be* (B) - constrained Pareto efficient *if there exist no collection of proportional and lump sum taxes/transfers* $((t_s)_{s\in\{H,L\}}, T)$ *such that the associated competitive equilibrium allocation constitutes a Pareto improvement.*

Proposition 19 *Competitive equilibria with linear prices (or bid-ask spreads) are, typically, (B) - constrained inefficient.*

[The idea is again to subsidize or tax purchases of insurance - at mized equilibria - so as to induce an increase in the level of trades by high wrt low types. Similarly, at the equilibria where high type don't trade, purchases of insurance should be subsidized]