Topics in Financial Economics

Piero Gottardi

Problem Set 1

Due: by April 16, 2018

Answers have to be submitted and prepared by each of you in isolation (though you are encouraged to discuss the problems and class material among you, what you submit should be your own answer)

1. (**arbitrage**) Consider a two period economy with $S = 3$ states at date 1. Suppose $J = 2$ assets are available for trade at $t = 0$ with returns:

$$r_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, r_2 = \begin{bmatrix} 8 \\ 2 \\ 10 \end{bmatrix}$$

   (a) Consider the prices $q_1 = 7, q_2 = 4$. Are there arbitrage possibilities at these prices?

   (b) Suppose a third asset is introduced, with returns $r_3' = [0, 0, 11]$ and price $q_3 = 5$. Are there arbitrage possibilities in this case? Discuss the results.

2. (**constrained efficiency**)Consider a two-period economy. There is uncertainty about the state of the world in the second period it is known that one of $s = 1, ..., S$ states will occur. The economy is populated by $h = 1, ..., H$ (types of) agents and 1 consumption good is available at each date. The consumption space is $X \subseteq \mathbb{R}_+^{S+1}$, agents are endowed with $\omega^h = \left(\omega_0^h, ., \omega_1^h(s), ..\right) \in \mathbb{R}_{++}^{S+1}$ $\forall h$ and have Von Neumann Morgenstern preferences $u^h(x_0) + \beta \sum_s \pi(s) u^h(x_1(s))$, where u^h is continuous, strongly monotonic, strictly concave.

   (a) Assume $J < S$ financial assets are available for trade at date 0, with returns $r_j(s)$, for $j = 1, .., J$, $s = 1, .., S$. There is no restriction on short sales. Define a constrained Pareto efficient allocation. Are the allocations obtained at a competitive equilibrium constrained Pareto efficient in this environment? Properly motivate your answer.

   (b) Assume now $J = S$ assets are available, with linearly independent returns (the matrix $R = [r_j(s)]_{j=1,..,J;s=1,..,S}$ of asset returns has full rank). However consumers face a limit on their short sales of assets, that is a trading constraint of the firm $\theta_j \geq \underline{\theta}$, for $\underline{\theta} < 0$, for all j.

      1. Assuming that u^h is also differentiable, write the first order conditions for a solution of the choice problem of a generic consumer h.

2. Are competitive equilibria Pareto efficient in this environment? And if not, are they constrained Pareto efficient? Provide a clear explanation for your answer.

3. (**efficient allocations under uncertainty**) Consider a three period exchange economy with a single consumption good. Time is denoted by $t = 0, 1, 2$; both at date $t = 1$ and $t = 2$ one out of $S = \{1, .., S\}$ states is realized. Each consumer $h = 1, .., H$ has an endowment/income of ω_0^h units at $t = 0$, and a random endowment $\widetilde{\omega}_1^h$ at date $t = 1$ and $\widetilde{\omega}_2^h$ at date $t = 2$. Assume $\widetilde{\omega}_1^h$, $\widetilde{\omega}_2^h$ are i.i.d. random variables, with finitely many (S) possible realizations. That is, the income process is i.i.d. over time for each consumer, but may well be correlated across different consumers. Let $s_1 \in S$ denote the state realized at $t = 1$ and $s_2 \in S$ the one realized at $t = 2$: the endowment of consumers of type h is then $\omega^h(s_1)$ at $t = 1$ and $\omega^h(s_2)$ at date $t = 2$. Consumer h has preferences described by the Von Neumann Morgenstern, time additive utility function:

$$u^h(x_0^h) + \beta \sum_{s_1 \in S} \pi(s_1) \left[u^h(x_1^h(s_1)) + \beta \sum_{s_2 \in S} \pi(s_2) u^h(x_2^h(s_1, s_2)) \right]$$

where $u^h(.)$ is strictly concave and differentiable and $x^h(s_1, s_2)$ is the consumption at date $t = 2$ when states $s_1 \in S$ and $s_2 \in S$ are realized respectively at $t = 1$ and $t = 2$. Note that probability beliefs $\pi(s_t)$ are the same across all consumers h. Denote by $\omega(s_t) = \sum_h \omega^h(s_t)$ the total resources available in state $s_t \in S$ at date t.

(a) Show that if $\mathbf{x}$ is a Pareto efficient allocation, then for all h, $x^h(s_1, s_2) = x^h(s_2)$ for all $s_1 \in S, s_2 \in S$ (that is, the allocation at date $t = 2$, only depends on the current realization of the uncertainty, not the previous one).

(b) Show that if $\mathbf{x}$ is a Pareto efficient allocation, then for any pair of states $s, s' \in S$ such that $\omega(s) \geq \omega(s')$, we have $x^h(s) \geq x^h(s')$ for all h, that is individual consumption varies co-monotonically with aggregate endowment.

(c) Show that the set of Pareto efficient allocations is invariant with respect to the value of the probability beliefs $\pi(s)$, $s \in S$.

(d) Do the results above (at (a), (b), (c)) hold if the beliefs are not common across agents?

4. (**comparative statics equilibrium asset prices**) Consider a two period economy with a single (representative) consumer with endowment ω_0 at $t = 0$ and $\omega_1(s)$ at $t = 1$ when state $s \in S$ is realized. The consumer's preferences are

$$u(x_0) + \beta \sum_{s \in S} \pi(s) u(x_1(s))$$

where $u(.)$ is strictly increasing and concave and such that $u''' > 0$. There are J assets (in zero net supply), which may be traded at $t = 0$, before the uncertainty is resolved. Denote by $r_j(s)$ the unit return of asset j at $t = 1$ when state $s \in S$ is realized and by q_j the corresponding price. Suppose asset $j = 1$ is riskless (that is, $r_1(s) = 1$ for all s).

(a) Find the competitive equilibrium price of a generic asset j.

(b) Consider the effect of the following change of the date 1 endowment, from $\omega(s)$ to $\omega'(s)$ where $\omega'(s), s \in S$ first order stochastically dominates the original endowment $\omega(s), s \in S$, i.e. the endowment at date 1 is higher. What will happen to the price of the riskless asset ($j = 1$)? Can you say anything about the prices of the other assets?

(c) Consider then the effect of another change, to $\omega''(s)$, where $\omega''(s), s \in S$ second order stochastically dominates the original endowment, i.e., the endowment is less risky. What will happen in this case to the price of the riskless asset? Can you say anything about the prices of the other assets?

5. (**constrained inefficiency**) Consider the model of a two period economy with production presented in class: there is a continuum of consumers, all with identical preferences $u(x_0) + \beta \sum_s \pi(s)u(x_1(s))$ and endowments, given by ω_0 units of he consumption good at date 0 and 1 unit of labor at date 1. Each consumer h can store an arbitrary amount $k > 0$ of the consumption good at date 0 to obtain a random amount $\tilde{\varsigma}^h k$ of capital at date 1, where $\tilde{\varsigma}^h$ is a random variable, i.i.d. across all consumers, with $\mathbb{E}\tilde{\varsigma}^h = 1$. In addition to consumers, there is a (representative) firm, operating at date 1 with a constant returns to scale technology $Y = F(K, L)$, to produce the consumption good using capital and labor as inputs. Let w and r denote the market price of labor and capital, at date 1.

(a) Write the conditions defining a competitive equilibrium for this economy (that is, the FOCs of the consumers' and the firm's optimization problem, and the market clearing conditions).

(b) Show that an (infinitesimal) change Δk of the investment decision of all consumers with respect to the level of the investment chosen in equilibrium improves the welfare of all consumers. What is the sign of Δk that improves welfare (that is, in equilibrium do we have over or under investment)? Explain the result. What does it tell us about the constrained efficiency of the competitive equilibrium?

6. (**equilibrium with production**) Consider a two period production economy, with (a large number of) $H = 2$ types of consumers and a large number of firms (all of the same type). Uncertainty is described by S states at date $t = 1$. Consumers have the same preferences $u(x_0) + \beta \sum_{s \in S} \pi(s)u(x_1(s))$ and the same endowments $\omega_0^h = \omega_0$ for $h = 1, 2$ at $t = 0$, but different endowments for $h = 1, 2 : \omega_1^1(s) = \omega(s) + \alpha z(s)$ and $\omega_1^2(s) = \omega(s) - \gamma z(s)$, at $t = 1$, for some $\alpha, \gamma > 0$, $z(s) > 0$ such that $\omega(s) + \alpha z(s) \geq 0$, $\omega(s) - \gamma z(s) \geq 0$, for $s = 1, .., S$. Each firm has the following production technology which allows to produce the consumption good at $t = 1$ using as input the consumption good at date 0 (each unit of consumption good invested at date 0 yields one unit of capital to be used as input at date 1) described by the production function: $y(s) = z(s)f(k)$, $s = 1, .., S$, $f' > 0, f'' < 0$. Note that the production technology is subject to shocks ($z(s)$).
The existing assets are a riskless bond and the firms' equity (that is a claim to the firms' date 1 cashflow); unlimited short sales of the existing assets *are allowed*. Each

consumers is initially endowed also with $\theta^h = 0.5$ shares of the representative firm. Suppose first that firms finance their period 0 investments only with equity (that is, initial shareholders are required to pay a fraction of the cost of the firm's investment equal to their share of the firm).

(a) Define a competitive equilibrium with production (as defined in class) for this economy.

(b) Write the first order conditions for a competitive equilibrium as defined (consider a symmetric equilibrium where all firms choose the same production plan). Examine whether at an equilibrium the willingness to pay for the equity of a firm (that is, their marginal utility of holding equity) is the same for the two types of consumers for the production plan chosen by the firm in equilibrium $(\bar{k}; \bar{y}(s) = z(s)f(\bar{k}), s = 1, .., S)$. And for any other production plan $(k; y(s) = z(s)f(k), s = 1, .., S)$?

(c) Show that Modigliani Miller Theorem holds in this set-up if firms' debt is riskless: that is, if firms were to finance their investment by issuing bonds (instead of using equity as above), their market value is unchanged.

(d) Show that if instead firms' debt is risky (that is, firms can default on their debt), Modigliani Miller Theorem does not hold, that is firms' market value may be affected if firms can issue risky bonds to fund at least part of their investment.