Dynamic Competitive Economies with Complete Markets and Collateral Constraints∗

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Abstract

In this paper we examine the competitive equilibria of a dynamic stochastic economy with complete markets and collateral constraints. We show that, provided both the set of asset payoffs and of collateral levels are sufficiently rich, the equilibrium allocations with sequential trades and collateral constraints are equivalent to those obtained in Arrow-Debreu markets subject to a series of appropriate limited pledgeability constraints.

We provide necessary and sufficient conditions for equilibria to be Pareto efficient and show that when collateral is scarce equilibria are not only Pareto inefficient but also often constrained inefficient, in the sense that imposing tighter borrowing restrictions can make everybody in the economy better off.

We derive sufficient conditions for the existence of Markov equilibria and, for the case of two agents, for the existence of equilibria which have finite support. These equilibria can be computed with arbitrary accuracy and the model is very tractable.

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1 Introduction

We examine the competitive equilibria of an infinite-horizon exchange economy where the only limit to risk sharing comes from the presence of a collateral constraint. Consumers face a borrowing limit, determined by the fact that all loans must be collateralized, as for example in Kiyotaki and Moore (1997) or Geanakoplos (1997), but otherwise financial markets are complete. Only part of the consumers’ future endowment can be pledged as collateral, hence the borrowing constraint may be binding and limit the risk sharing possibilities in the economy. More specifically, we consider an environment where consumers are unable to commit to repay their debt obligations and the seizure of the collateral by lenders is the only loss an agent faces for his default. There is no additional punishment, for instance in the form of exclusion from trade in financial markets as in the model considered by Kehoe and Levine (1993), (2001). However, like in that model, and in contrast to Bewley (1977) and the literature which followed it\(^\text{1}\), the level of the borrowing (and collateral) constraint is endogenously determined in equilibrium by the agents’ limited commitment problem.

The analysis is carried out in the set-up of a Lucas (1978) style economy with a single perishable consumption good. The part of a consumer’s endowment that can be pledged as collateral can be naturally interpreted as the agent’s initial share of the Lucas tree – a long-lived asset in positive supply that pays dividends at each date-event. This asset can be used, both directly and indirectly, as collateral for any short position of the consumer.

We show in this paper that this is a tractable model of dynamic economies under uncertainty, establish the existence of Markov and of finite support equilibria and analyze the welfare properties of competitive equilibria. More specifically, we show the equivalence between the competitive equilibria when trade occurs in a complete set of contingent commodity markets at the initial date, as in Arrow Debreu, subject to a series of appropriate limited pledgeability constraints, and the equilibria when trade is sequential, in a sufficiently rich set of financial markets, where short positions must be backed by collateral. This allows to clearly identify market structures, and in particular the specification of asset payoffs and of the associated collateral requirements, such that the only financial friction is the limited commitment requiring all loans to be collateralized. Second, we derive sufficient conditions for the existence of a Markov equilibrium in this model and show that Markov equilibria often have ‘finite support’ in the sense that individuals’ consumption only takes finitely many values. Markov equilibria exist whenever all agents’ coefficient of relative risk aversion is bounded above by one. Under the same assumption, or alternatively when all agents have identical, constant relative risk aversion utility functions, or when there is no aggregate uncertainty, equilibria have finite support if there are only two types of agents. Third, we provide some necessary and sufficient conditions for competitive equilibria to be fully Pareto efficient, that is for the amount of available collateral to be sufficiently large that the collateral constraint never binds. We then show that, whenever the constraint

\(^{1}\)See Heathcote, Storesletten and Violante (2009) for a survey.
binds, competitive equilibria in this model are not only Pareto inefficient but are also often constrained suboptimal, in the sense that introducing tighter restrictions on borrowing from some date \( t > 0 \) makes all agents better off.

Several papers (from the quoted work of Kiyotaki and Moore (1997), Geanakoplos (1997) to various others) have formalized the idea that borrowing on collateral might give rise to cyclical fluctuations in real activity and enhance the volatility of prices. They typically assume that financial markets are incomplete, and/or that the collateral requirements are exogenously specified, so that it is not clear if the source of the inefficiency are the missing markets or the limited ability of the agents to use the existing collateral for their borrowing needs. Furthermore, dynamic models with collateral constraints and incomplete markets turn out to be very difficult to analyze (see Kubler and Schmedders (2003) for a discussion), no conditions are known that ensure existence of recursive equilibria and there are therefore few quantitative results about the welfare losses due to collateral.

We show here that considering an environment where financial markets are complete and there are no restrictions to how the existing collateral can be used to back short positions allows to simplify matters considerably. In our model equilibria can often be characterized as the solution of a finite system of equation. We show that a numerical approximation of equilibria is fairly simple and a rigorous error analysis is possible. Moreover we can use the implicit function theorem to conduct local comparative statics and perform a serious quantitative analysis of the potential welfare gains from government intervention.

As mentioned above, there is also a large literature that assumes that agents can trade in complete financial markets, default is punished with the permanent exclusion from future trades and loans are not collateralized. We refer for convenience in what follows to these models as ‘limited enforcement models’. As shown in Kehoe and Levine (2001), Ligon et al. (2002) and Alvarez and Jermann (2000), these models are very tractable since competitive equilibria can be written as the solution to a planner’s problem subject to appropriate constraints. Even though this is not true in the environment considered here - the limited commitment constraint has a different nature and we show that competitive equilibria may be constrained inefficient - tractability still obtains.

Chien and Lustig (2010) (also Lustig (2000) in an earlier, similar work) examine a version of the model in this paper with a continuum of agents and growth. The main focus of their analysis is on a quantitative assessment of the asset pricing implications of the model and their similarities with Alvarez and Jermann (2000). Their notion of recursive equilibrium also uses instantaneous weights (Chien and Lustig call them ‘stochastic Pareto-Negishi weights’) as an endogenous state variable and is essentially identical to ours. However, our results on the existence of such recursive equilibria and of finite support equilibria are rather different, as explained more in detail in the next sections. Also, they do not examine how the allocation can be decentralized in asset markets with collateral constraints nor they discuss the constrained inefficiency of competitive equilibria.

Some related sufficient conditions for competitive equilibria with collateral constraints
to be Pareto efficient are derived by Cordoba (2008) in an environment with production, no aggregate uncertainty and a continuum of ex ante identical agents. For this case it can be verified that our conditions become equivalent to his.

Lorenzoni (2008), Kilenthong and Townsend (2011) as well as Gromb and Vayanos (2002) also show that collateral constraints can lead to constrained suboptimal equilibrium allocations. However, the analysis by Lorenzoni and Kilenthong and Townsend is different as they consider a production economy where capital accumulation links different periods and the reallocation is induced by a change in the level of investment that modifies available resources. In our pure exchange set-up resources are fixed, only their distribution can vary and the reallocation is induced by tightening the borrowing constraints with respect to their level endogenously determined in equilibrium. Gromb and Vayanos consider a model with segmented markets and competitive arbitrageurs which need to collateralize separately their positions in each asset, giving conditions under which reducing the arbitrageurs’ short positions in the initial period leads to a Pareto improvement. They also consider a pure exchange economy but the segmentation of markets is a key ingredient in their analysis.

Geanakoplos and Zame (2002, and, in a later version, 2009) are the first to formally introduce collateral constraints and default into general equilibrium models. They consider a two period model with incomplete markets where a durable good needs to be used as collateral. They are the first to point out that, even if markets are complete and the amount of collateral in the economy is large, the Pareto efficient Arrow Debreu allocation may not be obtained unless one allows for collateralized financial securities to be used as collateral in addition to the durable good (they refer to this as pyramiding). Our equivalence result in Section 2 below makes crucial use of this insight.

The remainder of the paper is organized as follows. In Section 2 we introduce the economic model and the equivalence of equilibrium allocations in three different market environments, with complete contingent markets at the initial date and with sequential trade in financial markets. In Section 3 we present a simple example to illustrate the properties of competitive equilibria. In Section 4 we study the existence of Markov equilibria and derive conditions under which they can be described by a finite system of equations. In Section 5 we analyze the welfare properties of equilibria. Most of the proofs are collected in the Appendix.

2 The model

We examine an infinite horizon stochastic model of an exchange economy with a single perishable consumption good available at each date $t = 0, 1, ...$. We represent the resolution of uncertainty by an event tree – at each period $t = 1, \ldots$ one of $S$ possible exogenous shocks $s \in S = \{1, \ldots, S\}$ occurs, with a fixed initial state $s_0 \in S$. Each node of the tree is characterized by a history of shocks $\sigma = s^t = (s_0, ..., s_t)$. The exogenous shocks follow a Markov process with transition matrix $\pi$, where $\pi(s, s')$ denotes the probability of shock $s'$
given $s$. We assume that $\pi(s, s') > 0$ for all $s, s' \in S$. With a slight abuse of notation we also write $\pi(s')$ to denote the unconditional probability of node $s'$. We collect all nodes of the infinite tree in a set $\Sigma$ and we write $\sigma' \succeq \sigma$ if node $\sigma'$ is either the same as node $\sigma$ or a (not necessarily immediate) successor. We write $\sigma' \succ \sigma$ if $\sigma'$ is a successor of (i.e. not the same as) $\sigma$.

There are $H$ infinitely lived agents which we collect in a set $\mathcal{H}$. Agent $h \in \mathcal{H}$ maximizes

$$U^h(c) = u^h(c_0, s_0) + E \left( \sum_{t=1}^{\infty} \beta^t u^h(c_t, s_t) \bigg| s_0 \right),$$

where (conditional) expectations are taken with respect to the Markov transition matrix $\pi$, and the discount factor $\beta \in (0, 1)$. We assume that the possibly state dependent Bernoulli function $u^h(\cdot, s) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is strictly monotone, $C^2$, strictly concave, and satisfies the Inada-condition $u^h'(c, s) = \frac{\partial u^h(c, s)}{\partial c} \rightarrow \infty$ as $c \rightarrow 0$, for all $s \in S$.

Each agent $h$’s endowment over his lifetime consists of two parts. The first part is given by an amount of the consumption good which the agent receives at any date event, i.e. $e^h(s_t) = e^h(s_t)$ where $e^h : S \rightarrow \mathbb{R}^+$ is a time-invariant function of the shock. In addition, the agent is endowed at period 0 with an exogenously given share $\theta^h(s^{-1}) \geq 0$ of a Lucas tree. The tree is an infinitely lived physical asset (can be interpreted as machines, land or houses), which pays each period strictly positive dividends $d : S \rightarrow \mathbb{R}^+$ that depend solely on the current shock realization $s \in S$. The tree exists in unit net supply, $\sum_{h \in \mathcal{H}} \theta^h(s^{-1}) = 1$, and its shares can be traded at any node $\sigma$ for a unit price $q(\sigma)$. The total endowment of the consumer is therefore $\omega^h(s_t) = e^h(s_t) + \theta^h(s^{-1})d(s_t)$, where $e^h(s_t)$ can be viewed as the non-pledgeable component, which cannot be sold in advance in order to finance consumption or savings at any date before the endowment is received.

To formalize the notion of complete markets with non-pledgeable labor-endowments, we define an **Arrow Debreu equilibrium with limited pledgeability** as a collection of prices $(\rho(\sigma))_{\sigma \in \Sigma}$ and a consumption allocation $(c^h(\sigma))_{h \in \mathcal{H}}$, $\sigma \in \Sigma$ such that

$$\sum_{h \in \mathcal{H}} (c^h(\sigma) - \omega^h(\sigma)) = 0, \quad \text{for all } \sigma \in \Sigma$$

and for all agents $h$

$$(c^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{c \geq 0} U^h(c) \text{ s.t.}$$

$$\sum_{\sigma \in \Sigma} \rho(\sigma)c(\sigma) \leq \sum_{\sigma \in \Sigma} \rho(\sigma)\omega^h(\sigma) < \infty$$

$$\sum_{\sigma \geq s^t} \rho(\sigma)c(\sigma) \geq \sum_{\sigma \geq s^t} \rho(\sigma)c^h(\sigma) \text{ for all } s^t.$$
markets, except for the additional constraints (4). These constraints express precisely the condition that \( e^h(\sigma) \) is unalienable, i.e. this component of the endowment can only be used to finance consumption in the node \( \sigma \) in which it is received or in any successor node. Note that these additional constraints are likely to be binding whenever the \( e^h \)-part of the agent’s endowments is large relative to the part given by the tree’s dividends, that is when there is only a small amount of future endowments that can be traded at earlier nodes of the event tree.

2.1 Financial markets with collateral constraints

We will show that the abstract equilibrium notion proposed above allows to capture the allocations attained as competitive equilibria in a standard setting where agents trade sequentially in financial markets, when short positions must be backed by collateral constraints, provided markets are ‘complete’ in a sense made precise below. This last feature is in contrast to many papers in the fairly large literature on models with collateral constraints, that assume incomplete financial markets. As we will see the notion of completeness of the market in this environment is a little subtler than usual, as it requires the presence of a sufficiently rich set of financial assets not only in terms of the specification of their payoff but also of their collateral requirements.

We consider an environment where at each node \( s^t \) any agent \( h \) can trade the tree as well as \( J \) financial assets (in zero net supply), collected in a set \( \mathcal{J} \). These assets are one-period securities: asset \( j \) traded at node \( s^t \) promises a payoff \( b_j(s^{t+1}) = b_j(s_{t+1}) \geq 0 \) at the \( S \) successor nodes \( (s^{t+1}) \). The agent can hold any amount \( \theta(s^t) \geq 0 \) of shares of the tree, which trade at the price \( q(s^t) \). In addition, for each security \( j \in \mathcal{J} \), with price \( p_j(s^t) \), the agent can hold any long-position \( \phi_j^+(s^t) \geq 0 \) as well as a short position \( \phi_j^-(s^t) \leq 0 \). The net position in security \( j \) is then denoted by \( \phi_j(s^t) = \phi_j^+(s^t) + \phi_j^-(s^t) \). We assume that all loans are non-recourse, that is consumers can default at no cost on the prescribed payments. To ensure that some payments are made, each short position in a security must be backed by an appropriate amount of the tree or of long-positions in other financial securities which are admissible as collateral. The specification of a financial security \( j \in \mathcal{J} \) is then given not only by its promised payoff \( b_j(.) \) but also by its collateral requirement, described by the vector\(^2 k_j^i \in \mathbb{R}^{J+1}_+ \). For each unit of security \( j \) sold short by a consumer, she is required to hold \( k_{j+1}^i \) units of the tree as well as \( k_j^i \) units of each security \( i \in \mathcal{J} \) as collateral. Since all loans are non-recourse, the consumer will find it optimal to default on her promise to deliver \( b_j(s_{t+1}) \) per unit sold whenever \( b_j(s_{t+1}) \) is higher than the value of the collateral associated with the short position. In this case the buyer of the financial security gets the collateral associated with the promise. Hence the actual payoff of any security \( j \in \mathcal{J} \) at any node \( s^{t+1} \) is endogenously determined by the agents’ incentives to default and the collateral.

\(^2\)In principle this collateral requirement could vary with the exogenous shock but for our purposes it suffices to assume that it is fixed.
requirements, as in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003). It is given by the values \( f_j(s^{t+1}) \) satisfying the following system of equations, for all \( j \in \mathcal{J} \):

\[
f_j(s^{t+1}) = \min \left\{ b_j(s_{t+1}), \sum_{i=1}^{J} k_j^i f_i(s^{t+1}) + k_{j+1}^l (q(s^{t+1}) + d(s_{t+1})) \right\}. \tag{5}
\]

For this equation to have a nontrivial solution, we assume that the tree is used as collateral for each security \( j \), either directly or indirectly. That is, if the tree is not used as collateral for security \( j \), i.e. \( k_{j+1}^l = 0 \), it must be used as collateral for some other security, in turn used as collateral for another security and so on until we reach one of the securities used as collateral for \( j \). In this way, the tree backs, indirectly, the claims of all securities along the chain. This construction will be made precise in the proof of Theorem 1.

A collateral constrained financial markets equilibrium is defined as a collection of choices \((c^h(\sigma), \theta^h(\sigma), (\phi^+_{i}(\sigma), \phi^-_{i}(\sigma)))_{\sigma \in \Sigma}\) for all agents \( h \in \mathcal{H} \), prices, \((p(\sigma), q(\sigma))_{\sigma \in \Sigma}\) and payoffs \((f(\sigma))_{\sigma \in \Sigma}\) satisfying (5) and the following other conditions:

(CC1) Market clearing:

\[
\sum_{h \in \mathcal{H}} \theta^h(\sigma) = 1 \quad \text{and} \quad \sum_{h \in \mathcal{H}} \phi^+_{i}(\sigma) + \sum_{h \in \mathcal{H}} \phi^-_{i}(\sigma) = 0 \quad \forall \sigma \in \Sigma.
\]

(CC2) Individual optimization: for each agent \( h \)

\[
(\theta^h(\sigma), \phi^+_{i}(\sigma), \phi^-_{i}(\sigma), c^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{\theta \geq 0, \phi_+ \geq 0, \phi_- \leq 0, c \geq 0} U^h(c) \quad \text{s.t.} \\
\theta(s^t) = c^h(s_t) + \phi(s^{t-1}) \cdot f(s^t) + \theta(s^{t-1})(q(s^t) + d(s_t)) - \theta(s^t)q(s^t) - \phi(s^t) \cdot p(s^t), \quad \forall s^t \\
\phi_j^+(s^t) + \sum_{j \in \mathcal{J}} k^j_{j+1}(s^t) \phi^-_{j}(s^t) \geq 0, \quad \forall s^t \\
\phi_j^-(s^t) + \sum_{i \in \mathcal{J}} k^i_j(s^t) \phi^-_{i}(s^t) \geq 0, \quad \forall s^t, \forall j \in \mathcal{J}.
\]

Existence of a collateral constrained financial markets equilibrium is proven in Kubler and Schmedders (2003).

While our assumptions on the rules governing collateral are obviously abstracting from many important issues arising in practice we try to model two key aspects of collateral contracts. First, we assume that margins are asset specific in that an asset cannot be used for two different short positions at the same time even if these two positions require payment in mutually exclusive states. It is important to point out that in this case no information over all the trades carried out by an agent is needed to enforce these collateral constraints - it suffices to post the required collateral for each short position; hence we can say that the financial contracts traded in the markets are non exclusive. This is in contrast with other limited commitment models, as Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000) where observability of all trades in financial markets is assumed. It is also in contrast to Chien and Lustig (2010) who assume that margins are portfolio-specific. They analyze
a model with collateral requirements where, in addition to the tree, a complete set of $S$
Arrow securities is available for trade at each node and the tree must be used as collateral
for short positions in these Arrow securities. Chien and Lustig assume that each unit of the
tree can be used to secure short positions in several Arrow securities at the same time, i.e.
the collateral constraint only has to hold for the whole portfolio of securities held. These
'portfolio margins' clearly allow to economize on the use of the tree as collateral but they
generally also require a stronger enforcement and coordination ability among lenders, or
the full observability of agents’ trades, not needed in the environment considered here. The
specification adopted here, based on asset specific margins, is closer to trading practices
used in financial markets (see e.g. Appendix A in Brunnermeier and Pedersen (2008) for
some details).

Second, the fact that margins are asset specific requires other channels to economize
on collateral. This is done via our assumption that not only the tree but also financial
securities can be used as collateral. Geanakoplos and Zame (2002) refer to this assumption as
'pyramiding'. In practice financial securities are routinely used for collateralized borrowing
(e.g. in repo agreement, see Bottazzi et al. (2012) but also in other transactions) – however
as Brunnermeier and Pedersen (2008) point out in order to take short positions in more
complicated securities such as derivatives brokers typically require cash-collateral.

Our assumption of pyramiding implies an implicit 're-use' of collateral that is somewhat
similar to 'rehypothecation', but there are some important differences. Rehypothecation
refers to the common practice in financial trades that allows a lender to use the collateral
he receives on a loan as collateral he pledges to enter a short-position with a third party.
In many collateralized trades the borrower remains the owner of the asset used as collateral
but the lender gains broad rights to use the collateral, in some trades the borrower loses
ownership over the pledged asset altogether (see e.g. Monnet (2011) for a description of
institutional details). We assume instead that a lender can reuse the collateral that is
backing his loan only indirectly by using the long position in the loan as collateral. Since
agents can default on their debt obligations, at the only cost of losing the posted collateral,
it is clear that the tree is ultimately backing all financial claims, directly or indirectly. But
the lender can never profit from a situation where the value of the collateral exceeds that
of the borrower’s obligations, by not returning the collateral. One possible reason why in
practice one sees rehypothecation rather than pyramiding is that financial securities are
only 'good collateral' if they are traded on liquid markets which might make it difficult
to build a large pyramid of financial securities with possibly different payoffs, backed by a
single physical asset.

2.2 Complete financial markets

We will show that any Arrow Debreu equilibrium allocation with limited pledgeability can
also be attained at a collateral constrained financial markets equilibrium, provided the asset
structure is 'sufficiently rich'. To see what this means in our environment, it is useful to first consider a simple 2-period example.

Suppose there are three agents with identical preferences trading in period 1 to insure against uncertainty in the second period. There are 3 equiprobable states in the second period (which in a slight abuse of notation we refer to as $s = 1, 2, 3$) and the tree pays 1 unit in each of these states. Each agent has initial holdings of the tree equal to 4 units, while the non-pledgeable second period endowment of the three agents is

$$e^1 = (0, 6, 9), \ e^2 = (6, 9, 0) \text{ and } e^3 = (9, 0, 6).$$

In this environment the Arrow Debreu equilibrium with limited pledgeability features a constant level of consumption in the three states and is then Pareto optimal and coincides with the standard Arrow Debreu equilibrium:

$$c^1 = c^2 = c^3 = (9, 9, 9).$$

It is easy to see that a complete set of Arrow securities, each of them collateralized by the tree, does not suffice to complete the market. To implement the above allocation in fact agent 1 would need to hold his endowment of the tree, buy 5 units of the Arrow security for state 1 and sell short, respectively, 1 and 4 units of the Arrow securities for states 2 and 3. However, this violates his collateral constraint – since he needs a total of 5 units of the tree as collateral while he only holds four units.

More interestingly if in addition to the Arrow securities there were also three assets paying zero in one state and one unit in the two others, each agent could achieve his Arrow-Debreu consumption without violating his collateral constraints by selling one unit of the asset that pays in the states where he has positive endowments (in addition to selling 3 units of the Arrow security that pays in the state where his endowment is 9). However no other agent would buy this asset since he needs to buy only one Arrow security to achieve his Arrow-Debreu consumption. Market clearing would not be possible. The same argument applies to all other specifications of the asset payoffs. In this example it is then not possible to achieve the Arrow Debreu equilibrium outcome if all promises are only backed by the tree.

In contrast, once one allows for pyramiding, i.e. for the presence of promises backed by financial securities and not the tree, one can easily find a set of asset trades that satisfy the collateral constraints and implement the Arrow-Debreu consumption allocation with limited pledgeability. Suppose there are two financial securities with promises $b_1 = (0, 1, 1), b_2 = (0, 0, 1)$. One units of the tree needs to be used as collateral for a short-position in security $j = 1$, while one unit of financial security 1 is used as collateral for each short-position in $j = 2$. Consider then the following portfolios. Agent 1 holds 9 units of the tree, $\theta^1 = 9, \theta^2 = 4, \theta^3 = 0$.

\footnote{It simplifies the exposition to assume that there are 12 trees in the economy. Alternatively, we could take the tree to be in unit supply and assume that it pays 12 units of dividends.}

\footnote{Kilenthong (2011) makes this point in a slightly different environment with capital.}
shorts 9 units of security 1, $\phi_{1-}^1 = -9$, using his holdings of the tree as collateral, and, at the same time, buys 3 units of this security back, $\phi_{1+}^1 = 3$. Finally, he shorts 3 units of security 2, $\phi_{2-}^1 = -3$, using the long position in security 1 as collateral. Agent 2 holds 3 units of the tree, shorts 3 units of security 1 and buys 9 units of security 2, $\theta^2 = 3, \phi_{1-}^2 = -3, \phi_{2+}^2 = 9$. Agent 3 holds no tree, buys 9 units of security 1 and shorts 6 units of security 2, backed by his holdings of security 1, i.e. $\theta^3 = 0, \phi_{1+}^3 = 9, \phi_{2-}^3 = -6$.

Note that given our specification of asset payoffs and collateral requirements, it is crucial that agent 1 can use a long-position in security 1 as collateral to back his short sales of security 2, although his net position in security 1 is negative; to this end, he holds at the same time long and short positions in the same security. Alternatively, we could have assumed that there are two distinct securities with payoff $(0, 0, 1)$, one of which is collateralized by the tree, the other, as above, by financial security $j = 1$.

The previous example illustrates the basic intuition of how to construct a set of assets which allows to attain the Arrow Debreu equilibrium allocations with limited pledgeability in a set-up with sequential trading of financial securities and collateral constraints. Our main result in this section generalizes this construction and the above argument to a dynamic economy with any number of states and consumers. To prove the result, it is convenient to introduce an alternative equilibrium notion with sequential trading, where each period intermediaries purchase the tree from consumers and issue on that basis, at no cost, a complete set of one period, state-contingent claims (options) on the tree, which are bought by consumers and are the only assets they can trade. This specification turns out to be very useful to analyze the properties of collateral constrained equilibria when markets are complete.

More precisely, at each node $s^t$ intermediaries purchase the tree and issue $J = S$ tree options, where asset $j$ promises the delivery of one unit of the tree the subsequent period if and only if shock $s = j$ realizes. Households in the economy can only take long positions in these assets at every node. The intermediaries’ holdings of the tree ensure that all due payments can be made. At any node $s^t$ an agent purchases a portfolio of $S$ tree-options $\theta_s(s^t) \geq 0, s = 1, \ldots, S$, at the prices $q_s(s^t) > 0, s = 1, \ldots, S$. The condition $q(s^t) = \sum_{s=1}^{S} q_s(s^t)$ ensures that intermediaries make zero profit in equilibrium, since the intermediation technology, with zero costs, exhibit constant returns to scale.

An equilibrium with intermediaries is defined as a collection of individual consumptions $(c_h(\sigma))_{h \in H}$, portfolios $(\theta^h(\sigma))_{h \in H, \sigma \in \Sigma}$, as well as prices $(q_s(\sigma))_{\sigma \in \Sigma, s \in S}$, such that markets clear and agents maximize their utility, i.e.

(IE1) At all nodes $s^t$,

$$\sum_{h \in H} \theta^h(s^t) = 1 \text{ for all } s \in S.$$
(IE2) For all agents $h \in \mathcal{H}$

$$(c^h, \theta^h) \in \arg \max_{\theta,c \geq 0} U^h(c) \text{ s.t.}$$

$$c(s^t) = c^h(s^t) + \theta_{s^t}(s^{t-1})\left(q(s^t) + d(s^t)\right) - \sum_{s'=1}^{S} \theta_{s^t}(s'q_{s^t}(s^t)) \text{ for all } s^t,$$

$$\theta_{s^t}(s^t) \geq 0, \text{ for all } s^t, s$$

It is relatively easy to show that any Arrow Debreu equilibrium allocation with limited pledgeability can also be attained as an equilibrium with intermediaries. In order to show that it can be attained as well as an equilibrium with collateral constraints, one needs to construct a rich enough asset structure that ensures that the payoffs achieved with the tree options can be replicated by trading in the asset market, subject to collateral constraints. The following theorem formalizes this.

**Theorem 1** *For any Arrow Debreu equilibrium with limited pledgeability there exists an equilibrium with intermediaries with the same consumption allocation. Moreover, there exists a sufficiently rich asset structure $J$ such that a collateral constrained financial markets equilibrium exists with the same consumption allocation.*

It is then easy to verify that the reverse implication also holds for collateral constrained equilibria without bubbles.\(^5\) Given this equivalence, in most of the paper we will consider the notion which turns out to be more convenient, depending on the issue the one of equilibrium with intermediaries or that of equilibrium with limited pledgeability.

Note that our collateral constrained equilibrium concept with complete financial markets has some interesting similarities to both Kehoe and Levine (1993) and Golosov and Tsyvinski (2007).\(^6\) Kehoe and Levine (1993) differs from most of the other papers in the literature on limited enforcement models for the fact that an environment with several physical commodities is considered. In the event of default only part of the agents’ endowment can be seized and agents also face the punishment of permanent exclusion from trade in financial markets, but their trades in the spot commodity markets are not observable and cannot be prevented. In addition to the inter-temporal budget constraint agents face then at each node a constraint on their continuation utility level, which in this case depends on (spot market) prices. Golosov and Tsyvinski (2007) consider an environment where insurance contracts are offered in the presence of moral hazard, but again hidden trades by the agents in some markets cannot be prevented and hence prices (together with agents’ utilities) again enter the agents’ incentive constraints. Prices also enter the additional constraint given by (4) above, which has however the form of a budget constraint (agents’ utilities do

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\(^5\)Since the existence proof in Kubler and Schmedders (2003) shows that collateral constrained financial markets equilibria without bubbles exist, Theorem 1 implies the existence also of Arrow Debreu equilibria with limited pledgeability.

\(^6\)We thank an anonymous referee for pointing out this connection to us.
not appear), and reflects the fact that no exclusion from trade in any market is possible. Agents’ incentives are captured by the specification of asset payoffs in (5) and trades at each node are always restricted by the collateral constraint. A possible interpretation of this is that agents can always hide all their trades: as argued in the previous section, no information over agents’ trades is needed to enforce the collateral constraints.

3 An example

To illustrate the analysis, we consider the simplest possible example, with two agents, two states and no aggregate uncertainty. The shocks are i.i.d. with two possible realizations, with probabilities \( \pi_1 = \pi(1, 1) = \pi(2, 1) \) and \( \pi_2 = \pi(1, 2) = \pi(2, 2) \). For simplicity, we assume the tree has a deterministic dividend \( d \) and the endowments of agent 1 are \( e^1(1) = h, e^1(2) = l \), the endowments of agent 2 are \( e^2(1) = l, e^2(2) = h \), where \( 0 < l < h \), and the agents’ Bernoulli utility function is state invariant, \( u^h(c, s) = u^h(c) \) for \( h = 1, 2 \).

We derive in what follows the competitive equilibria for different values of the parameters \( d, l \) and \( h \). We show for which values competitive equilibria are Pareto efficient and, when they are inefficient, we characterize their properties. Given the equivalence established in Theorem 1, we find it convenient to carry out the analysis here in terms of the notion of equilibrium with intermediaries.

3.1 Efficient equilibria

In the environment considered in this example there is no aggregate uncertainty and hence at a Pareto efficient equilibrium agents’ consumption is constant, i.e. we must have \( c^1(s^t) = \bar{c}^1 \) for all \( s^t \). This is also an equilibrium with intermediaries if there exist agents’ portfolios of tree options which support the efficient equilibrium allocation, given the equilibrium prices of the tree options, and satisfy the non-negativity constraints in (IE2).

The constant value of consumption implies that the supporting price of the tree is also constant and equal to

\[
\bar{q} = \frac{\beta d}{1 - \beta},
\]

while the prices of the state-contingent tree-options are \( q_{s'} = \pi_{s'} \beta (\bar{q} + d) \), invariant with respect to the past history as well as of the current shock realization \( s \) (since shocks are i.i.d.). It then follows from the above that \( q_{s'} = \pi_{s'} \bar{q} \) for all \( s' \). The supporting portfolios of tree options for agent 1 are obtained by substituting the above values of prices and consumption into the expressions of the budget constraint of this consumer when the shock realization is, respectively, 1 and 2:

\[
c^1 \quad = \quad h + \theta_1 (\bar{q} + d) - \bar{q}(\pi_1 \theta_1 + \pi_2 \theta_2) \quad (6)
\]

\[
= \quad l + \theta_2 (\bar{q} + d) - \bar{q}(\pi_1 \theta_1 + \pi_2 \theta_2), \quad (7)
\]
where \( \theta_2 \) is the amount held of the tree-option that pays in shock 2 and \( \theta_1 \) the amount held of the tree-option paying in shock 1 (both state invariant). Subtracting the second equation from the first one and substituting the value of \( \bar{q} \) obtained above yields

\[
\theta_2 - \theta_1 = \frac{(h - l)(1 - \beta)}{d}.
\]

By the market clearing conditions in the securities’ market, the holdings of tree options of type 2 consumers are then \((1 - \theta_1), (1 - \theta_2)\). The portfolio constraints, that is the non negativity of the securities’ holdings for both types of consumers, require that \( \theta_1, \theta_2 \in [0, 1] \), which means that \( \theta_1 \geq 0 \) and \( \theta_1 + (h - l)(1 - \beta)/d \leq 1 \). Thus there exists a value of \( \theta_1 \) satisfying these conditions if and only if

\[
\frac{(h - l)(1 - \beta)}{d} \leq 1.
\]  

If (8) holds, an efficient competitive equilibrium with intermediaries exists for some appropriate initial endowment of the tree, equal to the equilibrium portfolio of tree options, for the type 1 consumers \( \theta^1(s_{-1}) = \theta_1 \geq 0 \) when \( s_0 = 1 \) and \( \theta^1(s_{-1}) = \theta_2 = \frac{(h - l)(1 - \beta)}{d} + \theta_1 = \frac{(h - l) + \theta_1 (\bar{q} + d)}{\bar{q} + d} \leq 1 \) when \( s_0 = 2 \). The equilibrium consumption level of type 1 consumers is then obtained by substituting these values into the budget constraints (6).

### 3.2 Steady state equilibria

If condition (8) is not satisfied, a Pareto efficient competitive equilibrium with intermediaries does not exist, so that the only possible equilibrium is an inefficient one, where the constraints on agents’ portfolios bind (at least in some state). Assume that \( l > 0 \). We show that in the environment of this simple example a steady state\(^7\) equilibrium exists and is supported by the following steady state portfolios\(^8\) that are independent of the shock

\[
\theta^1 = (0, 1), \theta^2 = (1, 0).
\]

Letting \( q_1(s) \) and \( q_2(s) \) denote the equilibrium prices of the tree options, which now depends on the current state \( s \), the consumption values of agent 1 supported by the above portfolios readily obtain from the budget constraints:

\[
c^1(1) = h - q_2(1),
\]

\[
c^1(2) = l + (d + q_1(2) + q_2(2)) - q_2(2) = l + d + q_1(2)
\]

The values of the equilibrium prices must satisfy the first order conditions of agent 1 for the security paying in state 2 (since agent 1 is always unconstrained in his holdings of this asset)

\[
q_2(1)u^{1'}(c^1(1)) = \beta \pi_2(q_1(2) + q_2(2) + d)u^{1'}(c^1(2))
\]

\[
q_2(2)u^{1'}(c^1(2)) = \beta \pi_2(q_1(2) + q_2(2) + d)u^{1'}(c^1(2))
\]

\(^7\)We use the term steady state to refer to situations where the equilibrium variables depend at most on the current realization of the shock.

\(^8\)Note that these are the portfolios supporting an efficient equilibrium if \( h - l = d/(1 - \beta) \).
and, by the same argument, the corresponding conditions of agent 2 for the security paying in state 1

\[ q_1(1)u^{2'}(c^2(1)) = \beta \pi_1 (q_1(1) + q_2(1) + \delta)u^{2'}(c^2(1)) \]
\[ q_1(2)u^{2'}(c^2(2)) = \beta \pi_1 (q_1(1) + q_2(1) + \delta)u^{2'}(c^2(1)) \]

From the second and the third conditions above we obtain that the following relationship must hold

\[ q_1(1) = \frac{\beta \pi_1}{1 - \beta \pi_1}(q_2(1) + \delta) \]
\[ q_2(2) = \frac{\beta \pi_2}{1 - \beta \pi_2}(q_1(2) + \delta) \]

To complete the proof that a steady state equilibrium exists with the portfolio-holdings stated above we establish the following lemma.

**Lemma 1** The remaining first order conditions

\[ q_2(1)u^1(h - q_2(1)) = \frac{\beta \pi_2}{1 - \beta \pi_2}(q_1(2) + \delta)u^1(1 + \delta + q_1(2)) \] (9)
\[ q_1(2)u^2(h - q_1(2)) = \frac{\beta \pi_1}{1 - \beta \pi_1}(q_2(1) + \delta)u^2(1 + \delta + q_2(1)) \]

have a solution for a positive level of the prices \( q_2(1), q_1(2) \) satisfying \( h - q_2(1) \geq 1 + \delta + q_1(2) \).

### 4 Stationary equilibria

The example demonstrates that in the environment considered, even when the constraints bind there can exist equilibria where consumption and prices only depend on the current realization of the exogenous shock. While this is generally the case for pure exchange economies with Pareto optimal equilibria, in models with incomplete markets, or with overlapping generations, equilibrium prices and consumption levels typically take infinitely many values along an equilibrium path. It is obviously an important question whether along the equilibrium path the endogenous variables take finitely many or infinitely many values. If they take finitely many values, the equilibrium can be characterized by a finite system of equations, it can typically be computed easily and one can conduct local comparative statics using the implicit function theorem. In this case, we say that there exists a finite-support equilibrium since the stochastic process of the exogenous and endogenous variables has finite support. Ligon et al. (2002) show that in limited enforcement models finite support equilibria always exist if there are two agents. However, in those models equilibrium allocations are constrained efficient and can be obtained as the solution of a convex programming problem. As shown in what follows, competitive equilibrium in our model may be constrained inefficient and it is not possible to derive equilibrium allocations as the solution
to a planner’s problem – the argument in Ligon et al. (2002) crucially depends on this property. In this respect our model is closer to models with incomplete financial markets and in these models finite support equilibria typically do not exist.

Even if equilibria have infinite support they might still be tractable if they are Markov for some simple, endogenous state variable (as for example is the case in the stochastic growth model). In many models with heterogeneous agents and market imperfections, however, it is an open problem under which conditions Markov equilibria exist, i.e. whether prices and consumption are a function of the current shock and some other state variable, as the beginning-of-period wealth distribution (see e.g. Kubler and Schmedders (2002) or Santos (2002)).

In this section we discuss both the existence of equilibria with finite support and of Markov equilibria. We begin by providing some sufficient conditions for the existence of Markov equilibria. This analysis becomes simpler if we consider the Arrow Debreu equilibrium notion with limited pledgeability and use as endogenous state variable the instantaneous Negishi-weights, yielding current consumption levels (as in Chien and Lustig (2010)), rather than the beginning of period distribution of the tree. The intuition for why this simplifies the analysis is that, as we will see below, if at some node an agent’s limited pledgeability constraint does not bind, the agent’s instantaneous Negishi-weight remains constant - hence one only needs to analyze the evolution of the weight for nodes where the constraints are binding. In the remainder of the section we show that, for economies with only two types of agents, finite support equilibria exist under more general conditions and conclude with a brief discussion of the existence of finite support equilibria with more than two agents.

4.1 Markov equilibria

We take the endogenous state at some node $s^t$ to be the Negishi consumption weights $\lambda(s^t) \in \mathbb{R}_+^H$ where

$$(c^1(s^t), \ldots, c^H(s^t)) \in \arg \max_{c \in \mathbb{R}_+^H} \sum_{h \in H} \lambda^h(s^t)u^h(c^h, s^t) \text{ s.t. } \sum_{h \in H} (c^h - \omega^h(s^t)) = 0.$$ 

Negishi’s (1960) approach to proving existence of a competitive equilibrium, instead of solving for consumption values that clear markets, solves for weights that enforce budget balance (see also Dana (1993)). Judd et al. (2003) show how to use this approach to compute equilibria in Lucas style models with complete markets (and without collateral constraints). Chien and Lustig (2010) (see also Chien et al. (2011)) consider a Markov equilibrium notion that features individual multipliers - interpretable as the inverse of our consumption weights - as endogenous state variable in a model with collateral constraints analogous to ours, though for a slightly different economy, with a continuum of agents.

The state then consists of the current shock and all agents’ current Negishi weights, $(s, \lambda)$. To define a Markov equilibrium, we need to specify a policy function that deter-
mines how the endogenous variables depend on the state and a transition that maps the current state to a probability distribution over next period’s states. The consumption policy function, \( C : \mathcal{S} \times \mathbb{R}^H_+ \rightarrow \mathbb{R}^H_+ \) is obviously given by

\[
C(s, \lambda) = \arg \max_{c \in \mathbb{R}^H_+} \sum_{h \in \mathcal{H}} \lambda^h u^h(c^h, s) \text{ s.t. } \sum_{h \in \mathcal{H}} (c^h - \omega^h(s)) = 0. 
\tag{10}
\]

To understand how \( \lambda \) evolves across time periods and shock realizations, consider an Arrow Debreu equilibrium with limited pledgeability with prices \( (\rho(\sigma))_{\sigma \in \Sigma} \) and a consumption allocation \( (c^h(\sigma))_{h \in \mathcal{H}}(\sigma \in \Sigma) \). If for an agent \( h \) and a node \( s^t \) the limited-pledgeability-constraint does not bind, i.e.

\[
\sum_{\sigma \geq s^t} \rho(\sigma)c^h(\sigma) > \sum_{\sigma \geq s^t} \rho(\sigma)e^h(\sigma),
\]

his marginal rate of substitution between \( s^{t-1} \) and \( s^t \) must equal the price ratio, \( \frac{\rho(s^1)}{\rho(s^{t-1})} \) and, as we show formally below, we have \( \lambda^h(s^t) = \lambda^h(s^{t-1}) \). If, on the other hand, the constraint binds

\[
\sum_{\sigma \geq s^t} \rho(\sigma)c^h(\sigma) = \sum_{\sigma \geq s^t} \rho(\sigma)e^h(\sigma),
\]

his marginal rate of substitution must be higher than \( \frac{\rho(s^1)}{\rho(s^{t-1})} \) and we have \( \lambda^h(s^t) > \lambda^h(s^{t-1}) \).

At a recursive equilibrium we need to write recursively the value of an agents’ future lifetime consumption in excess of his endowments. We denote this ‘excess expenditure function’ by \( V^h(s, \lambda) \). As in the Negishi approach, we take an agent’s marginal utility to value this consumption. Intuitively this is possible because, as argued above, whenever the agent is unconstrained at some state \( s \) his marginal rate of substitution between \( s \) and the predecessor state \( s' \) equals the prices while, when he is constrained, we have \( V^h(s', \lambda) = 0 \) and it is irrelevant whether this term is multiplied by the agents’ marginal rate of substitution between \( s \) and \( s' \) or by actual market prices since the product is always zero. Given the role of the agents’ marginal utilities in the definition of the functions \( V^h(s, \lambda) \) it is useful to define (in a slight abuse of notation)

\[
u^{h'}(s, \lambda) = u^{h'}\left(C^h(s, \lambda), s\right).
\]

A recursive equilibrium (or Markov equilibrium) then consists of a policy function \( C : \mathcal{S} \times \mathbb{R}^H_+ \rightarrow \mathbb{R}^H_+ \) together with a transition function \( L : \mathcal{S} \times \mathbb{R}^H_+ \rightarrow \mathbb{R}^H_+ \) and excess expenditure functions \( V^h : \mathcal{S} \times \mathbb{R}^H_+ \rightarrow \mathbb{R} \) for all agents \( h \in \mathcal{H} \) such that for all \( h \in \mathcal{H} \), all \( s \in \mathcal{S} \) and all \( \lambda \in \mathbb{R}^H_+ \)

\[
V^h(s, \lambda) = u^{h'}(s, \lambda) \left(C^h(s, \lambda) - e^h(s)\right) + \beta \sum_{s'} \pi(s, s')V^h(s', L(s', \lambda)) \tag{11}
\]

and for all \( s' \in \mathcal{S} \),

\[
V(s', L(s', \lambda)) \geq 0 \tag{12}
\]

\[
L(s', \lambda) - \lambda \geq 0 \tag{13}
\]

\[
V(s', L(s', \lambda)) \left(L(s', \lambda) - \lambda\right) = 0 \tag{14}
\]
The following result formalizes the fact that recursive equilibria are Arrow-Debreu equilibria with limited pledgeability.

**Theorem 2** Given a recursive equilibrium \((C, V, L)\) and any \(\lambda_0 \in R_{++}^H\) with \(V^h(s_0, \lambda_0) \geq 0\) for all \(h\), there exist initial tree-holdings \((\theta^h(s^{-1}))_{h \in H}\) and an Arrow Debreu equilibrium with limited pledgeability with \(c^h(s^t) = C^h(s_t, \lambda(s^t))\) and \(\lambda(s^t) = L(s_t, \lambda(s^{t-1}))\) for all \(s^t, t > 0\).

Note that if there is a competitive equilibrium with \(\lambda(s^t) = \lambda^*\) for all \(s^t\), this must be an unconstrained Arrow Debreu equilibrium. The fact that \(\lambda(s^t)\) does not change over time implies that the additional constraint (4) is never binding in equilibrium and the allocation is identical to the unconstrained Arrow-Debreu equilibrium allocation and is Pareto-optimal. Therefore, if for a given Markov equilibrium a vector of weights \(\lambda^*\) exists with \(V^h(s, \lambda^*) \geq 0\) for all \(s \in S\) and all \(h \in H\), then there exist initial conditions (corresponding to the weights \(\lambda^*\)) for which the Markov equilibrium is identical to an unconstrained Arrow-Debreu equilibrium.

It is well known that in models where the equilibrium may be constrained inefficient Markov equilibria might not always exist. For the model with collateral constraints, when financial markets are incomplete no sufficient conditions are known that ensure the existence of a Markov equilibrium (see Kubler and Schmedders (2003)). In contrast, in the environment considered here, with complete markets, as shown in the next proposition, the assumption that all agents’ preferences satisfy the gross substitute property implies that Markov equilibria always exist. Dana (1993) shows that this assumption guarantees the uniqueness of Arrow-Debreu equilibria in infinite horizon exchange economies without constraints. We show that her argument extends to our model and guarantees the existence of a recursive equilibrium through the uniqueness of the ‘continuation-equilibrium’. As pointed out by Dana (1993), in our context the assumption of gross substitutes is equivalent to assuming that for all agents \(h\) and all shocks \(s\), the term \(c u^h(c, s)\) is increasing in \(c\); or equivalently, that the coefficient of relative risk aversion \(-c u^h''(c, s) u^h(c, s)\) is always less than or equal to one. While in applied work it is often assumed that relative risk aversion is significantly above one it is also sometimes argued (see e.g. Boldrin and Levine (2001)) that a value below one might be the empirically more relevant case.

We have then the following result.

**Theorem 3** Suppose that for all agents \(h\) and all shocks \(s\), \(c u^h(c, s)\) is increasing in \(c\) for all \(c > 0\). Then a Markov equilibrium exists and it is unique.

### 4.2 Markov equilibria with finite support

The main difficulty in determining whether Markov equilibria with finite support exist lies in specifying the support. We show that for the case of two agents’ types, \(H = 2\), there is a natural characterization of the support. We will extend then the analysis to the case of arbitrarily many agents’ types.
4.2.1 Finite support Markov equilibria in economies with two types of agents

When there are only two types of agents finite support equilibria exist under more general conditions than those of Theorem 1 and are extremely easy to characterize. Clearly, since the functions \( C(s,.) \) and \( V(s,.) \) are homogeneous of degree 0 we can normalize \((\lambda^1, \lambda^2)\) to always lie in the unit interval. This allows us to denote by \( \lambda = \lambda^1 \) the value of the consumption weight for agent 1 and take this as a state variable. In a slight abuse of notation, we write \( C^h(s, \lambda) = C^h(s, (\lambda, 1 - \lambda)) \), \( V^h(s, \lambda) = V^h(s, (\lambda, 1 - \lambda)) \) etc.

In order to fix ideas, suppose that a Markov equilibrium exists and that, for all agents \( h = 1, 2 \) and all \( s \in S \), the excess expenditure function \( V^h(s,.) \), has a unique zero. Denote by \( \lambda^*(s) \) the zero of \( V^1(s,.) \) and by \( \lambda^*(s) > \lambda^*(s) \) the zero of \( V^2(s,.) \).\(^9\) Figure 1 illustrates a possible form of these functions and their zeros in the simple case where there are two possible shocks, \( s = 1, 2 \). By Equations (12)-(14), for each \((s, \lambda) \in S \times (0,1)\) we must have either \( L(s, \lambda) = \lambda \) (if \( V^h(s, \lambda) \geq 0 \) for \( h = 1, 2 \)) or \( L(s, \lambda) \in \{ \lambda^*(s), \lambda^*(s) \} \) (if \( V^1(s, \lambda) < 0 \), by (12) we must 'jump' to \( \lambda^*(s) \) and if \( V^2(s, \lambda) < 0 \) we 'jump' to \( \lambda^*(s) \)). Therefore, if each \( V^h(s,.) \) has a unique zero the entire equilibrium transition can be described by the 2S numbers \( \lambda^*(1), \lambda^*(1), ..., \lambda^*(S), \lambda^*(S) \), either \( \lambda \) stays constant or it 'jumps' to one of these 2S values. In Figure 1 if we start with \( \lambda = \lambda^*(1) \) when the current state is \( s = 1 \), the endogenous state has to move to \( \lambda^*(2) \) when state 2 occurs and will then alternate (as in Ligon et al (2002)) between the values \( \lambda^*(1) \) and \( \lambda^*(2) \).

Unfortunately, it is not straightforward to identify conditions under which, when a Markov equilibrium exists, each excess expenditure function \( V^h(s,.) \) has a unique zero. In what follows we adopt so a constructive approach and conjecture that a competitive equilibrium with finite support exists. We derive then a finite system of equations and inequalities that characterize the competitive equilibrium in this case and find conditions under which this system has a solution.

We conjecture in particular that a Markov equilibrium exists where at most 2S points are visited in the endogenous state space. We denote them by \( (\lambda^*(s), \lambda^*(s))_{s \in S} \). For any \( \lambda \in (0,1)^S \) and \( \lambda \in (0,1)^S \) we define a function \( L : S \times [0,1] \rightarrow [0,1] \) by

\[
L(\lambda^*, \lambda^*)(s, \lambda) = \begin{cases} 
\lambda & \text{ if } \lambda^*(s) \leq \lambda \leq \lambda^*(s) \\
\lambda^*(s) & \text{ if } \lambda < \lambda^*(s) \\
\lambda^*(s) & \text{ if } \lambda > \lambda^*(s).
\end{cases}
\]

This function \( L(\lambda^*, \lambda^*)(., .) \) describes a law of motion for \( \lambda \). When \( \lambda = \lambda^* \) and \( \lambda = \lambda^* \), \( L(\lambda^*, \lambda^*)(., .) \) is then the transition function for a competitive equilibrium.

To characterize when this is the case define, for each \( h = 1, 2 \), 2S numbers \( V^h(s, \lambda^*(s)) \),
V^h(s, \bar{\lambda}(\tilde{s})) for s, \tilde{s} \in S to be the solution of the following linear system of \(2S^2\) equations:

\[
V^h(s, \bar{\lambda}(\tilde{s})) = u^{h'}(s, \bar{\lambda}(\tilde{s})) \left( C^h(s, \bar{\lambda}(\tilde{s})) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L(\bar{\lambda}, \bar{\lambda})(s', \bar{\lambda}(\tilde{s}))), \tag{15}
\]

\[
V^h(s, \bar{\lambda}(\tilde{s})) = u^{h'}(s, \bar{\lambda}(\tilde{s})) \left( C^h(s, \bar{\lambda}(\tilde{s})) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L(\bar{\lambda}, \bar{\lambda})(s', \bar{\lambda}(\tilde{s}))). \tag{16}
\]

The equilibrium values \((\bar{\lambda}(s), \bar{\lambda}(s))_{s \in S} = (\bar{\lambda}^*(s), \bar{\lambda}^*(s))_{s \in S}\) must then satisfy the following conditions:

\[
V^1(s, \bar{\lambda}^*(s)) = V^2(s, \bar{\lambda}^*(s)) = 0 \text{ for all } s \in S, \tag{17}
\]

as well as, for all \(s, s'\)

\[
L(\bar{\lambda}^*, \bar{\lambda}^*)(s', \bar{\lambda}^*(s)) = \bar{\lambda}^*(s) \Rightarrow V^h(s', \bar{\lambda}^*(s)) \geq 0 \text{ for } h = 1, 2 \tag{18}
\]

\[
L(\bar{\lambda}^*, \bar{\lambda}^*)(s', \bar{\lambda}^*(s)) = \bar{\lambda}^*(s) \Rightarrow V^h(s', \bar{\lambda}^*(s)) \geq 0 \text{ for } h = 1, 2 \tag{19}
\]

When the above conditions are satisfied, by construction, \(L(\bar{\lambda}^*, \bar{\lambda}^*)(.)\) describes a transition function that ensures that \(V^h(s, L(s, \lambda)) \geq 0\) for all \(\lambda \in \{\bar{\lambda}^*(1), \ldots, \bar{\lambda}^*(S)\}\) so that (12) holds. Moreover by construction (17) ensures that there exists a homogenous transformation of \((\lambda^1, \lambda^2)\) (dropping the normalization \(\lambda^1 + \lambda^2 = 1\)) such that (13) and (14) must hold as well. Hence the transition \(L(\bar{\lambda}^*, \bar{\lambda}^*)(s, .)\) describes a competitive equilibrium.

To prove existence we first show that there always exist \(S\) pairs \((\bar{\lambda}^*(s), \bar{\lambda}^*(s))\) such that the solution to (15) and (16) satisfy (17):

**Lemma 2** There always exists \(\bar{\lambda}^*(1), \ldots, \bar{\lambda}^*(S) \in (0, 1)^2S\) solving (15), (16) and (17).

Next we discuss conditions that ensure that (18) and (19) hold. Unlike for the previous lemma, to establish this result we need to make some fairly strong assumptions on preferences or endowments. Given a solution to system (15)-(17) we can define functions \(V^h : S \times (0, 1) \rightarrow \mathbb{R}, h = 1, 2\), as follows.

\[
V^1(s, \lambda) = u^1(s, \lambda)(C^1(s, \lambda) - e^1(s)) + \beta \sum_{s'} \pi(s, s') V^1(s', L(\bar{\lambda}^*, \bar{\lambda}^*)(s, \lambda))
\]

\[
V^2(s, \lambda) = u^2(s, \lambda)(C^2(s, \lambda) - e^2(s)) + \beta \sum_{s'} \pi(s, s') V^2(s', L(\bar{\lambda}^*, \bar{\lambda}^*)(s, \lambda))
\]

A sufficient condition for (18) and (19) to hold is that each \(V^h(s, .)\) has a unique zero. When this is the case, for both \(h = 1, 2\), \(V^h(s, \lambda)\) is non-negative for all \(\lambda \in [\bar{\lambda}^*, \bar{\lambda}^*]\) since \(V^1(s, \lambda) > 0\) for all \(\lambda > \bar{\lambda}(s)\) and \(V^2(s, \lambda) > 0\) for all \(\lambda < \bar{\lambda}(s)\) - the case we discussed at the beginning of this section. To guarantee this we make the following strong assumption on preferences:

**Assumption 1** The preferences of all agents satisfy one of the following properties:
1. The coefficient of relative risk aversion $-cu^{hl}(c, s)/u^{hl}(c, s) \leq 1$ for all $c, s, h$.

2. All agents have identical, constant relative risk aversion (CRRA) Bernoulli utility functions.

3. $u^h(c, s)$ is state independent for all $h$ and there is no aggregate uncertainty.

If all agents’ relative risk aversion is below or equal to 1, the utility satisfies the gross substitute property and the result follows from the proof of Theorem 3, since we showed uniqueness of Markov equilibria. In order to prove the sufficiency of conditions 2. and 3., it is useful to define the following functions $V^h(s, \lambda) = \frac{1}{u^h(s, \lambda)}V^h(s, \lambda)$ for $h = 1, 2$. Clearly $V^h(s, \lambda)$ has a unique zero if and only if $\hat{V}^h(s, \lambda)$ does. We have

$$\hat{V}^1(s, \lambda) = C^1(s, \lambda) - c^1(s) + \beta \sum_{s': \lambda \in [\lambda^*(s'), \lambda^*(s)]} \pi(s, s') \frac{u^{11}(s', \lambda)}{u^{11}(s, \lambda)} \hat{V}^1(s', \lambda) + \beta \sum_{s': \lambda > \lambda^*(s')} \pi(s, s') \frac{u^{11}(s', \lambda)}{u^{11}(s, \lambda)} \hat{V}^1(s', \lambda, \lambda^*(s'))$$

(20)

$$\hat{V}^2(s, \lambda) = C^2(s, \lambda) - c^2(s) + \beta \sum_{s': \lambda \in [\lambda^*(s'), \lambda^*(s)]} \pi(s, s') \frac{u^{21}(s', \lambda)}{u^{21}(s, \lambda)} \hat{V}^2(s', \lambda) + \beta \sum_{s': \lambda < \lambda^*(s')} \pi(s, s') \frac{u^{21}(s', \lambda)}{u^{21}(s, \lambda)} \hat{V}^2(s', \lambda, \lambda^*(s'))$$

(21)

for all $s$ with $\lambda \in [\lambda^*(s'), \lambda^*(s')]$.

Assume that agents have identical CRRA preferences. Then the term $u^{hl}(s', \lambda)/u^{hl}(s, \lambda)$ is independent of $\lambda$. Therefore $\lambda$ only enters $\hat{V}^1(s, \lambda)$ through the term $C^1(s, \lambda)$, which is clearly increasing in $\lambda$, and through the term $\pi(s, s') \frac{u^{11}(s', \lambda)}{u^{11}(s, \lambda)} \hat{V}^1(s', \lambda, \lambda^*(s'))$, which is also increasing in $\lambda$ since $u^{hl}(s, \lambda)$ is decreasing in $\lambda$. Therefore the function $\hat{V}^1$ must be monotonically increasing and has a unique zero. Finally, if there is no aggregate uncertainty, the term $u^{hl}(s', \lambda)/u^{hl}(s, \lambda)$ is simply equal to 1 and the same argument as for identical CRRA preferences shows the monotonicity of $V^h(s, \lambda)$.

The previous arguments establish that Assumption 1 guarantees the existence of a Markov equilibrium with finite support:

**Theorem 4** Under any one of the conditions of Assumption 1 a finite support Markov equilibrium exists in economies with two types of agents.

Note that in the above construction the intervals $([\lambda^*(s), \lambda^*(s)])_{s \in S}$ uniquely define the values $(V^h(s, \lambda(s)), V^h(s, \lambda(s)))_{s \in H(s)}$ and characterize the equilibrium. The equilibrium dynamics of these equilibria is then straightforward. If one starts at an initial condition

\[\text{[In an earlier working paper version of their published paper, Chien and Lustig also characterize equilibria with finite support for the case of two shocks and two agents with identical CRRA utility. Our result holds for any number of shocks under more general conditions.]}\]
which corresponds to a welfare weight on the boundary of the interval $[\lambda^*(s), \bar{\lambda}^*(s)]$ for the initial state $s = s_0$, only finitely many different welfare weights are visited along the equilibrium. Moreover, since (18) and (19) constitute a finite number of inequalities, it is possible to verify numerically whether a Markov equilibrium exists. This is an important advantage of finite support equilibria; if equilibria have infinite support it is often extremely difficult to conduct error analysis given a computed approximate Markov equilibrium.

Ligon et al. (2002) establish an analogous result for two agent economies with limited enforcement, where equilibria are solutions of a constrained planner’s problem. In that model, because the planner’s problem can be formulated as a stationary programming problem, Markov equilibria always exist and to establish the finite support result it suffices to ensure the monotonicity of the agents’ indirect utility function. In our model the existence of a Markov equilibrium is not guaranteed and even if Markov equilibria exist we need to ensure the monotonicity of the expenditure function. Hence the conditions in Ligon et al. (2002) are weaker than the ones stated in Assumption 1.

4.2.2 The example again

To illustrate the construction of Theorem 4 it is useful to reconsider the example of Section 3. In that example a Pareto-efficient steady state exists, for some parameter values, but, depending on the initial conditions, it might take arbitrarily long to reach it. This can be easily explained in this framework. Suppose for simplicity that $h - l = d_1 - \beta$, $u^1(c) = u^2(c) = \log(c)$ and $\pi_1 = \pi_2 = \frac{1}{2}$. Denote aggregate endowments by $\omega = h + l + d$. It is easy to check that with log-utility we have $C^1(s, \lambda) = \lambda \omega$ for both $s = 1, 2$. Also, by the argument in Section 3.1, there exists a unique efficient steady state where each agents’ consumption is given by $\frac{\omega}{2}$. As pointed out after Theorem 2, a Pareto efficient Markov equilibrium exists (for some initial conditions) if, for some $\lambda^*$, we have $V^h(s, \lambda^*) \geq 0$ for all $h$ and all $s$. In the environment considered here, since agent 1 has a high endowment in shock 1, we must have $\lambda^*(1) > \lambda^*(2)$ and, for an efficient equilibrium to exist, we must also have $\lambda^*(1) \leq \bar{\lambda}^*(2)$. In fact, we will show that an efficient Markov equilibrium exists with $\lambda^*(1) = \bar{\lambda}^*(2) = \frac{1}{2}$. For this, one needs to verify that, with this value of $\lambda^*(1)$, the solution to the following system (obtained from (15), (16), with $L(s, \lambda^*(1)) = \lambda^*(1)$ for $s = 1, 2$)

\[
V^1(1, \lambda^*(1)) = 1 - \frac{h}{\lambda^*(1)\omega} + \frac{\beta}{2} \left( V^1(1, \lambda^*(1)) + V^1(2, \lambda^*(1)) \right)
\]

\[
V^1(2, \lambda^*(1)) = 1 - \frac{l}{\lambda^*(1)\omega} + \frac{\beta}{2} \left( V^1(1, \lambda^*(1)) + V^1(2, \lambda^*(1)) \right)
\]

satisfies $V^1(1, \lambda^*(1)) = 0$ and $V^1(2, \lambda^*(1)) > 0$. It is easy to see that if $V^1(1, \lambda^*(1)) = 0$ a solution of the second equation above, when $\lambda^*(1) = \frac{1}{2}$, is given by $V^1(2, \lambda^*(1)) = \frac{1}{1-\beta/2} (1 - \frac{1}{\beta \omega^*})$, always positive. Substituting this value into the first equation yields

\[
V^1(1, \lambda^*(1)) = 1 - \frac{h}{0.5\omega} + \frac{\beta}{2} V^1(2, \lambda^*(1)) = \frac{l + d - h}{\omega} + \frac{\beta (h + d - l)}{(2-\beta)\omega} = 0,
\]
which holds whenever \((2 - \beta)(l + d - h) + \beta(h + d - l) = 0\), equivalent to our assumption that \(h - l = \frac{d}{1 - \beta}\).

By symmetry, \(\lambda^*(2) = \frac{1}{2}\) solves the corresponding system for \(V^2(s, \lambda^*(2))\), \(s = 1, 2\). We can also solve for \(\lambda^*(2)\) the system for \(V^1(s, \lambda^*(2))\), \(s = 1, 2\), where \(L(1, \lambda^*(2)) = \lambda^*(1)\). Using the fact that \(V^1(1, \lambda^*(1)) = 0\) we get

\[
V^1(2, \lambda^*(2)) = 0 = 1 - \frac{l}{\lambda^*(2)\omega},
\]

thus \(\lambda^*(2) = \frac{1}{\omega} < 1/2\). By symmetry, we have that \(\lambda^*(1) = 1 - \lambda^*(2)\).

As pointed out at the end of the previous section, the values \(\lambda^*(s), \lambda^*(s), s = 1, 2\) completely characterize the Markov equilibrium for this example. If the initial conditions are such that the initial consumption weight \(\lambda^*(s_0) = 1/2\), the Markov equilibrium coincides with the efficient steady state. On the other hand if, for example, \(\lambda^*(s_0) = \lambda^*(2)\), the state variable remains unchanged at the initial value \(\lambda^*(s_0)\) as long as only shock 2 occurs, since \(V^h(2, \lambda^*(2)) \geq 0\) for both \(h = 1, 2\). Agent 1 will consume then an amount less than 1/2. When shock 1 occurs, we have \(V^1(1, \lambda^*(2)) < 0\) since \(\lambda^*(2) < \lambda^*(1)\) and \(V^1(1, .)\) is increasing. Therefore \(\lambda\) must ‘jump’ to \(\lambda^*(1)\) where it will stay from there on. Hence the steady state will be reached after each shock has realized at least once.

The same argument can also be used to analyze the case where the steady state is inefficient. It is easy to see that when \(h - l > \frac{d}{1 - \beta}\) we have \(\lambda^*(1) > \lambda^*(2)\). There is then no efficient steady state and along the equilibrium path the instantaneous Negishi weight \(\lambda\) oscillates between the two values \(\{\lambda^*(2), \lambda^*(1)\}\).

### 4.2.3 Existence and non-existence of finite support equilibria when \(H > 2\)

Unfortunately, for the general case with more than two agents’ types we do not know of general conditions which ensure the existence of finite support Markov equilibria. The problem is that the dynamics of the Negishi weights, which as we saw has a simple pattern when \(H = 2\), can be much more complex when \(H > 2\). In fact, even for limited enforcement models no existence results of finite support equilibria are available when \(H > 2\).

However, it is easy to construct examples for which finite support equilibria exist. Suppose there are 3 types of agents and three equiprobable i.i.d. shocks. Assume again the agents have identical log-utility functions, \(u^h(c) = \log(c)\) for all \(h\), endowments are \(e^1 = (0, h, h)\), \(e^2 = (h, 0, h)\) and \(e^3 = (h, h, 0)\) for some \(h > 0\), while the tree pays constant dividends \(d > 0\). The aggregate endowment is then deterministic and equal to \(\omega = 2h + d\). Similarly to the previous example, we have \(C^h(s, \lambda) = \lambda^h\omega\) for all \(h\) and \(s\). We assume that initial conditions are \(s_0 = 1\) and \(\theta^1(s_{-1}) = 1\).

Using symmetry, we show in what follows that under the condition

\[
h > \frac{d}{1 - \beta}
\]
there exists a steady state where agents 2 and 3 are constrained in state 1, agents 1 and 3 in state 2 and agents 1 and 2 in state 3. Denoting by $\lambda(s, h)$ the value of the Negishi weights in state $s$ where only type $h$ is unconstrained, we need then to find the values of the vectors $\lambda(1, 1), \lambda(2, 2)$ and $\lambda(3, 3)$ constituting the support of the equilibrium. By symmetry, the weights of all agents when constrained are identical, across all states, i.e. $\lambda^1(2, 2) = \lambda^3(3, 3) = \lambda^2(1, 1) = \lambda^3(2, 2) = \lambda_h$ for some $\lambda_h$. Similarly, $\lambda^1(1, 1) = \lambda^2(2, 2) = \lambda^3(3, 3) = \lambda_l = 1 - 2\lambda_h$. In this situation, the transition function must then satisfy the following property

$$L(s, \lambda) = \lambda(s, s)$$

whenever $\lambda \in \{\lambda(1, 1), \lambda(2, 2), \lambda(3, 3)\}$.

Given this property of the transition function and the above specification of the states where each agent is constrained, proceeding analogously to the previous section we obtain

$$V^1(1, \lambda(1, 1)) = 1 + \frac{\beta}{3} V^1(1, \lambda(1, 1)) = \frac{1}{1 - \frac{\beta}{3}}$$

$$V^1(s, \lambda(s, s)) = 0 = 1 - \frac{h}{\lambda^1(s, s)} + \frac{\beta}{3} V^1(1, \lambda(1, 1)), \ s \neq 2, 3$$

where the equality $V^1(s, \lambda(s, s)) = 0$ holds in the states where agent 1 is constrained. Hence we must have

$$1 - \frac{h}{\lambda_h} + \frac{\beta}{3 - \beta} = 0,$$

or

$$\lambda_h = \frac{(3 - \beta)h}{3(3 + 2h)},$$

and $1 > \lambda_h \geq \frac{3}{9} = \lambda_l$ given the assumption $h \geq \frac{1}{1 - \beta}$. Given the initial condition, we must have $\lambda(s_0) = \lambda(1, 1)$ and we have verified the one constructed is a Markov equilibrium with finite support.

To illustrate why it is difficult to find general conditions that ensure the existence of a finite support equilibrium, consider the following small modification of the example above. Instead of assuming that each agents’ individual endowments are high in two out of the three states, suppose they are high only in one out of the three states. That is $e^1 = (h, 0, 0), e^2 = (0, h, 0)$ and $e^3 = (0, 0, h)$ for some $h > 0$. Under the maintained assumption of logarithmic utility, Theorem 3 ensures the existence of a unique Markov equilibrium. However, we will show that under the condition $h > \frac{1}{2(1 - \beta)}$ the unique Markov equilibrium has infinite support (and hence there exists no finite support equilibrium).

It is clear from the specification of the agents’ endowments that now in equilibrium two out of the three agents’s types will always be unconstrained. Consider the case where the current state of the economy is $(s, \lambda)$ for $s \neq 1$, and we move next period to shock 1 and weight $\lambda'$, where agent 1 is constrained ($V^1(1, \lambda') = 0$). Hence agents 2 and 3 will be unconstrained and the ratio $\lambda^2/\lambda^3$ will be equal to $\lambda^2/\lambda^3$, and depend so on the previous period’s state.
In fact, $\lambda$ must satisfy the following system of equations, for some function $L(s, \cdot)$,

$$V^1(1, \lambda) = 0 = 1 - \frac{h}{\lambda^1\omega} + \frac{\beta}{3} \sum_{s' = 2}^{3} V^1(s', L(s', \lambda))$$

$$V^1(s, \lambda) = 1 + \frac{\beta}{3} \sum_{s' = 2}^{3} V^1(s', L(s', \lambda)), \quad s = 2, 3.$$

Since for $s = 2, 3$ we obtain that $V^1(s, \lambda) = \frac{1}{1 - \frac{2\beta}{3}}, \lambda^1 = \lambda_h$ is determined as solution of the following equation.

$$1 - \frac{h}{\lambda_h\omega} + \frac{2\beta}{3 - 2\beta} = 0 \iff \lambda_h = \frac{h}{\omega} \frac{3 - 2\beta}{3}.$$

Under the assumption that $2h > \frac{3}{1 - \beta}$ we have $1 > \lambda_h > \frac{1}{3}$.

By symmetry we conjecture that the same expression obtains for agent 2 in state 2 and agent 3 in state 3. We therefore postulate the following transition function, defined on the set of all $\lambda \in \Delta^2$ with $\lambda^j = \lambda_h$ for some $j = 1, 2, 3$:

$$L(s, \lambda) = \begin{cases} (\lambda^{j'})^H \text{ if } j = s \\ \lambda^{j'} = \lambda_h \text{ otherwise.} \end{cases}$$

If we take as initial conditions $s_0 = 1$ and $\theta^2(s_{-1}) = \theta^3(s_{-1}) = 1/2$, a Markov equilibrium exists with transition function $L(s, \cdot)$. The initial value of the welfare weights is given by $\lambda(s_0) = (\lambda_h, \frac{1 - \lambda_h}{2}, \frac{1 - \lambda_h}{2})$. Since $V^h(s, \lambda) > 0$ whenever $s \neq h$ and, by construction, for all $\lambda(s^t)$ along the equilibrium path $V^h(h, \lambda(s^t)) = 0$, the one specified is a Markov equilibrium. It is easy to check that this equilibrium generally does not have finite support. To see this, consider for instance a sequence of shocks for $t = 1, 2, ...$ with $s_t = 1$ if $t$ is odd and $s_t = 2$ if $t$ is even. It is easy to see that we must have $\lambda^3(s^{t+1}) = \lambda^3(s^t) \frac{1 - \lambda_h}{\lambda_h + \lambda^3(s^t)}$ and hence

$$\frac{1}{\lambda^3(s^{t+1})} = \frac{1}{1 - \lambda_h} + \frac{\lambda_h}{1 - \lambda_h} \frac{1}{\lambda^3(s^t)},$$

which converges to $\frac{1}{1 - 2\lambda_h}$ if $\lambda_h < \frac{1}{2}$ or diverges otherwise. In the process it takes infinitely many values.

5 Welfare Properties

In this section we investigate the welfare properties of competitive equilibria with collateral constraints. We have seen in the example considered in Section 3, that equilibria are Pareto efficient whenever $d/(1 - \beta) \geq h - 1$. Here we generalize this result and derive some necessary and sufficient conditions for the existence of Pareto efficient equilibria in general economies with no aggregate uncertainty as well as with aggregate uncertainty when consumers have identical CRRA preferences. These conditions, similarly to the one derived for the example, require the amount of available collateral to be sufficiently large relative to the variability of agents’ endowments.
Next, we will turn our attention to the welfare properties of equilibria when the collateral constraint binds so that competitive equilibria are Pareto inefficient. We compute the (wealth-equivalent) welfare losses for a class of general and realistic economies. In addition, we show that when competitive equilibria are not Pareto efficient, they are also constrained inefficient. That is, even by taking the limited pledgeability constraints into account, a welfare improvement can still be obtained with respect to the competitive equilibrium.

5.1 Pareto Efficient equilibria

It is well known that in the stationary economy considered in this paper Pareto efficient allocations are always stationary, i.e. consumption only depends on the current shock (see e.g. Judd et al. (2003)). As in Section 3.2 we define a steady state to be an equilibrium where individual consumption and prices are time invariant functions of the shock alone. Therefore a competitive equilibrium can only be Pareto efficient if it is a steady state equilibrium.

As shown in the example of Section 3, even when an efficient steady state exists, for some initial conditions it may not be reached immediately and in that case the equilibrium is still inefficient. With a slight abuse of notation, we then say that Pareto efficient equilibria exist if there are initial conditions for which the competitive equilibrium is Pareto efficient (and therefore is a steady state equilibrium that is immediately reached).

Consider then a Pareto efficient allocation \( \{ c^h(s) \}_{h \in H, s \in S} \). For this allocation to be supported as an Arrow-Debreu equilibrium with limited pledgeability the supporting prices, given by \( \rho(s^t) = u^{ht}(c^h(s^t), s^t) \beta^t \pi(s^t) \) for all \( s^t \) and any \( h \), must be such that the limited pledgeability constraints are satisfied for all agents \( h \in H \) and all shocks \( s \in S \):

\[
u^{ht}(c^h(s), s)(c^h(s) - e^h(s)) + E \left( \sum_{t=1}^{\infty} \beta^t u^{ht}(c^h(s^t), s^t)(c^h(s^t) - e^h(s^t)) \bigg| s_0 = s \right) \geq 0 \quad (22)
\]

and the inter-temporal budget constraint (3) also holds given the initial endowment distribution. For general utility functions and endowments, this is a difficult problem. However, we identify below environments where the problem reduces to a simple condition that is easy to verify.

5.1.1 No aggregate uncertainty

When there is no aggregate uncertainty, i.e. \( \sum_{h \in H} \omega^h(s) \) is equal to a constant \( \omega \) for all shock realizations \( s \in S \), and agents’ Bernoulli functions are state independent, Pareto-efficient allocations are such that consumption is also constant, \( c^h(s) = c^h \) for all \( s, h \). Hence condition (22) simplifies to

\[
\max_{s \in S} \left[ c^h(s) + E \left( \sum_{t=1}^{\infty} \beta^t e^h(s^t) \bigg| s_0 = s \right) \right] \leq \frac{c^h}{1 - \beta} \quad \text{for all } h \in H \quad (23)
\]
and, using the feasibility of the allocation we obtain the following:\textsuperscript{11}

\textbf{Theorem 5} A necessary and sufficient condition for the existence of a Pareto-efficient equilibrium with no aggregate uncertainty is

\begin{equation}
(1 - \beta) \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \left[ e^h(s) + E \left( \sum_{t=1}^{\infty} \beta^t e^h(s_t) \middle| s_0 = s \right) \right] \leq \omega. \tag{24}
\end{equation}

Recalling that $\omega - \sum_{h \in \mathcal{H}} e^h(s) = d(s)$, condition (24) requires the amount of collateral in every state, measured by $d(s)$, to be sufficiently large relative to the variability of the present discounted value of the agents’ non-pledgeable endowment, captured by the term on the left hand side of (24).

It is useful to consider how the above condition simplifies if in addition shocks are i.i.d. and $d(s) = d$ for all $s$. In this case $E \left( \sum_{t=1}^{\infty} \beta^t e^h(s_t) \middle| s_0 = s \right)$ is independent of $s$ and hence (24) can be rewritten as

$$
\sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} e^h(s) \leq \frac{\omega - \beta e}{1 - \beta} = \frac{d}{1 - \beta} + e,
$$

where $e = \sum_{h \in \mathcal{H}} e^h(s)$ for any $s$.\textsuperscript{12}

\subsection{5.1.2 Identical CRRA preferences}

Consider next the case where all agents have identical CRRA preferences with coefficient of relative risk aversion $\tau$. All Pareto-efficient allocations satisfy then the property that, for all $h, s$, $e^h(s) = \lambda^h \omega(s)$ for some $\lambda^h \geq 0$ and $\sum_{h \in \mathcal{H}} \lambda^h = 1$, where $\omega(s) = \sum_{h \in \mathcal{H}} \omega^h(s)$. We can therefore write condition (22) as

$$
\frac{\lambda^h}{\omega(s)^{\tau-1}} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{\lambda^h}{\omega(s_t)^{\tau-1}} \middle| s_0 = s \right) = \frac{e^h(s)}{\omega(s)^{\tau-1}} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)^{\tau-1}} \middle| s_0 = s \right)
$$

for all $s \in \mathcal{S}$, $h \in \mathcal{H}$.

As in the previous section, feasibility allows us to obtain from the above inequality the following necessary and sufficient condition for the existence of an efficient equilibrium:

\begin{equation}
1 \geq \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \frac{e^h(s)}{\omega(s)^{\tau-1}} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)^{\tau-1}} \middle| s_0 = s \right)
\end{equation}

when all agents have log-utility, i.e. $\tau = 1$, condition (25) greatly simplifies and reduces to

$$
\frac{1}{1 - \beta} \geq \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \left[ \frac{e^h(s)}{\omega(s)} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)} \middle| s_0 = s \right) \right],
$$

\textsuperscript{11}Necessity is obvious given (23). Sufficiency then follows from the observation that under (24) it is always possible to find a Pareto efficient allocation $(e^h)_{h \in \mathcal{H}}$ that satisfies (23).

\textsuperscript{12}It is easy to verify that in the setting of the example of Section 3, this condition is equivalent to

$$
\beta - 1 \leq \frac{d}{1 - \beta}. 
$$
analogous to (24).

Condition (25), while not very intuitive, can obviously be verified numerically for given processes of individual endowments and dividends. It is obviously beyond the scope of this paper to take a stand on which values should be considered as 'realistic' for the level of persistence and the size of the idiosyncratic shocks as well as for the amount of available collateral. It might be interesting, however, to consider an example of a calibrated economy from the applied literature. Heaton and Lucas (1996) calibrate a Lucas style economy with two types of agents to match key facts in the US economy. They take the 'dividend-share' to be earnings to stock-market capital and estimate this number to be around 15 percent of total income. They assume that aggregate growth rates follow an 8-state Markov chain and calibrate their model using the PSID (Panel Study of Income Dynamics) and NIPA (National Income and Product Accounts). Let us consider their calibration for the ‘Cyclical Distribution Case’ but de-trend the economy to ensure we remain in our stationary environment. We find that for their specification of the economy the competitive equilibrium is Pareto efficient in the long run, i.e. condition (25) holds. This shows that, if one considers the specification of idiosyncratic risks in Heaton and Lucas (1996) to be somewhat realistic, Pareto efficiency obtains in the long run for all realistic levels of collateral.

5.2 Constrained inefficiency

If the collateral in the economy is too little to support a Pareto efficient allocation, it could still be the case that the equilibrium allocation is constrained Pareto efficient in the sense that no reallocation of the resources that is feasible and satisfies the collateral constraints can make everybody better off. We show here that this is not true, by presenting a robust example for which a welfare improvement can indeed be found subject to these constraints.

We consider in particular a reallocation obtained by imposing tighter short-sale constraints on the trades of the tree options in an equilibrium with intermediaries; in the light of the equivalence established in 2.2 this is equivalent to increasing the collateral requirements in a collateral constrained equilibrium. We consider then the associated equilibrium where agents optimize subject to such constraints and markets clear. Such reallocation clearly respects the collateral constraints. At the same time, since the tighter constraints will change trades and hence securities’ prices, the allocation obtained may not be budget feasible at the original prices and looser short-sale constraints, and hence might yield a higher welfare.

We show the result in the simple environment described in Section 3, when shocks are equiprobable, $\pi_1 = \pi_2 = \frac{1}{2}$, consumers have the same preferences, $u^h(c, s) = u(c)$ for $h = 1, 2$, and their endowment in the low state is zero: $l = 0$. In addition, $h(1 - \beta) > \delta$, so that (8) is violated and there is no Pareto efficient equilibrium, but an inefficient steady state equilibrium exists.

Suppose the economy is at this inefficient steady state, where $\theta_1 = (0, 1)$, $\theta_2 = (1, 0)$,
and consider the welfare effect of tightening the portfolio restriction to \( \theta^h_s(s') \geq \varepsilon \), for \( \varepsilon > 0 \) and all \( s \in S \). The restriction is assumed to be introduced at \( t = 1 \) and to hold for all \( t \geq 1 \). The intervention is announced at \( t = 0 \) after all trades have taken place. Agents’ utility is then evaluated ex ante, from date 0. We show that this intervention is Pareto improving, for an open set of the parameter values describing the economy. Thus the inefficient steady state equilibrium is also constrained inefficient: making the collateral constraint tighter in some date events improves welfare.

Given the nature of the intervention and the fact that the economy is initially in a steady state, there is a transition phase of one period before the economy settles to a new steady state\(^{13}\): prices and allocations are then going to depend now on time (whether it is \( t = 1 \) or \( t > 1 \)) as well as the realization of the current shock. It is convenient to use the notation \( q_{s'}(s; t) \) to indicate the price at time \( t \) and state \( s \) of the tree option that pays in state \( s' \). As before, the price of the tree is then \( q(s; t) = q_1(s; t) + q_2(s; t) \). The new equilibrium portfolios are, at all dates \( t \geq 1 \), \( \theta^1 = (\varepsilon, 1 - \varepsilon) \), \( \theta^2 = (1 - \varepsilon, \varepsilon) \), that is the short-sale constraint always binds. At the date of the intervention, \( t = 1 \), we have then

\[
\begin{align*}
    c^1(s_1 = 1) &= h - q_1(1; 1)\varepsilon - q_2(1; 1)(1 - \varepsilon) \\
    c^1(s_1 = 2) &= d + q_1(2; 1) + q_2(2; 1) - q_1(2; 1)\varepsilon - q_2(2; 1)(1 - \varepsilon)
\end{align*}
\]

At all subsequent dates, \( t > 1 \),

\[
\begin{align*}
    c^1(s_t = 1) &= h + \varepsilon(q_1(1) + q_2(1) + d) - q_1(1)\varepsilon - q_2(1)(1 - \varepsilon) \\
    &= h + \varepsilon d - q_2(1)(1 - 2\varepsilon) \\
    c^1(s_t = 2) &= (d + q_1(2) + q_2(2))(1 - \varepsilon) - q_1(2)\varepsilon - q_2(2)(1 - \varepsilon) \\
    &= d(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)
\end{align*}
\]

That is, we settle at the new steady state where \( q_{s'}(s; t) = q_{s'}(s) \) for all \( t > 1, s, s' \).

The above expressions allow us to gain some intuition for the effects of the intervention considered. Consider first the direct effect, ignoring the price changes: we see that the intervention unambiguously increases the variability of consumption across states not only at all dates \( t > 1 \), but also at \( t = 1 \).\(^{14}\) Turning next our attention to the price changes, we show in what follows that the equilibrium price of the tree options unambiguously increases, as a result of the intervention, since their effective supply (the amount which can be traded in the market) decreases, from \( 1 \) to \( 1 - 2\varepsilon \). From the above expressions we see that an

\[^{13}\text{Note that this is different from what we found in Section 4.2.2, where we showed that the transition to a steady state may take a very long time, until all shocks occurred. The reason is that the original steady state is no longer feasible when the restriction \( \theta^h_s(s') \geq \varepsilon \) is introduced, and hence it is no longer possible for the allocation to stay the same until the shock stays the same, as before.}\]

\[^{14}\text{This last property follows from the fact that the consumer’s optimality conditions imply that, at a steady state equilibrium, we have } q_2(1) > q_1(1) \text{ and } q_1(2) > q_2(2).\]
increase in prices reduces the variability of consumption across states, since consumers are net buyers of assets when they are rich and net sellers when they are poor. Hence the price effect improves risk sharing, in contrast to the direct effect. We also show that for an open set of parameter values the price effect prevails over the direct effect.

We have eight new equilibrium prices to determine. By symmetry (of consumers’ preferences, endowments and shocks) however these reduce to four, since \( q_1(1;1) = q_2(2;1) \), \( q_2(1;1) = q_1(2;1) \), as well as \( q_1(1) = q_2(2) \) and \( q_2(1) = q_1(2) \) for all \( t = 2, \ldots \). Using the above expressions of the budget constraints, the equilibrium prices can be obtained from the first order conditions for the consumers’ optimal choices. After some substitutions, we obtain\(^{15}\) the following equation that can be solved for \( q_2(1) = q_1(2) \):

\[
q_1(2)u'(h + \varepsilon d - q_1(2)(1 - 2\varepsilon)) - \frac{\beta(q_1(2) + \delta)}{2 - \beta} u'(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon) = 0. \tag{26}
\]

It is useful to denote by \( q_1^0(2) \) the solution of this equation when \( \varepsilon = 0 \) (that is, at the initial steady state). Differentiating (26) with respect to \( \varepsilon \), and evaluating it at \( \varepsilon = 0 \) yields the following expression:

\[
\frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0} = -\frac{\beta q_1^0(2) u''(\delta) + q_1^0(2)u''(h)}{u'(h) - u'(\delta) - \beta \frac{\delta + q_1^0(2)}{2 - \beta} u''(\delta) - q_1^0(2) u''(h)} \tag{27}
\]

where \( u'(h) = u'(h - q_1^0(2)) \) and \( u'(\delta) = u'(\delta + q_1^0(2)) \) with \( u''(h) \) and \( u''(\delta) \) defined analogously. In the above expression the numerator is clearly positive, and so is the denominator, since equation (26) evaluated at \( \varepsilon = 0 \) yields \( u'(h) = \frac{\beta q_1^0(2)}{2 - \beta} u'(\delta) > \frac{\beta}{2 - \beta} u'(\delta) \). From the above expressions of the budget constraints and the symmetry of equilibrium prices we find that the effect on equilibrium consumption in the new steady state is

\[
\frac{dc^1(s_t = 1)}{d\varepsilon} \bigg|_{\varepsilon=0} = -\frac{dc^1(s_t = 2)}{d\varepsilon} \bigg|_{\varepsilon=0} = 2q_1^0(2) + \delta - \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0}. \tag{28}
\]

From (27) we immediately see that

\[
0 < \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0} < \delta + 2q_1^0(2),
\]

so that \( \frac{dc^1(s_t=1)}{d\varepsilon} \bigg|_{\varepsilon=0} > 0 \). Hence in the new steady state, that is for all \( t > 1 \), the equilibrium price of the tree options unambiguously increases, as claimed, as a result of the intervention, but the change in prices is not enough to overturn the direct effect of the intervention, and so the variability in consumption across states increases too.

We can similarly proceed to determine the effect on consumption at the transition date \( t = 1 \):

\[
\frac{dc^1(s_1 = 1)}{d\varepsilon} \bigg|_{\varepsilon=0} = -\frac{dc^1(s_1 = 2)}{d\varepsilon} \bigg|_{\varepsilon=0} = q_1^0(2) - \frac{\beta(q_1^0(2) + \delta)}{2 - \beta} - \frac{dq_2(1;1)}{d\varepsilon} \bigg|_{\varepsilon=0},
\]

\(^{15}\)The details for this as well as the similar derivation of (31) below are in the Appendix.
where we used the fact that \( q_1(1;1) \), evaluated at \( \varepsilon = 0 \), equals \( q_1(1) \) and both terms are at the steady state value before the intervention, \( \frac{\beta(q_1(2)+d\beta)}{2-\beta} \).

The effect on the discounted expected utility of consumer 1 of an infinitesimal tightening of the portfolio restriction, that is from \( \varepsilon = 0 \) to \( d\varepsilon > 0 \) is then

\[
\frac{dU}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{1}{2} \left[ u'(h) - u'(0) \right] \frac{dc^1(s_1 = 1)}{d\varepsilon} \bigg|_{\varepsilon=0} + \frac{\beta}{2(1-\beta)} \left[ u'(h) - u'(0) \right] \frac{dc^1(s_t = 1)}{d\varepsilon} \bigg|_{\varepsilon=0}.
\]

(29)

By symmetry, the expression for the change in consumer 2’s expected utility has the same value. Hence the welfare effect of the intervention considered is determined by the sign of the expression in (29).

Since \( u'(h) < u'(0) \), our finding on the sign of (28) implies that the effect of the intervention considered on agents’ steady state welfare, given by the second term in (29), is always negative. For the intervention to be welfare improving we need then to have a welfare improvement in the initial period that is sufficiently large to compensate for the negative effect after that period. More precisely, from (29) it follows that \( \frac{dU}{d\varepsilon} \bigg|_{\varepsilon=0} > 0 \) if, and only if,

\[
\frac{dc^1(s_1 = 1)}{d\varepsilon} \bigg|_{\varepsilon=0} < -\frac{\beta}{1-\beta} \frac{dc^1(s_t = 1)}{d\varepsilon} \bigg|_{\varepsilon=0},
\]

or equivalently, substituting the expressions obtained above for the consumption changes and rearranging terms,

\[
\frac{dq_2(1;1)}{d\varepsilon} \bigg|_{\varepsilon=0} > \frac{2q_1^0(2) + \delta\beta}{(2-\beta)(1-\beta)} - \frac{\beta}{1-\beta} \frac{dq_2(1)}{d\varepsilon} \bigg|_{\varepsilon=0}.
\]

(30)

That is, for an improvement to obtain the price change in the first period, \( \frac{dq_2(1;1)}{d\varepsilon} \bigg|_{\varepsilon=0} \), has to be sufficiently large so that \( c^1(s_1 = 1) \) decreases, increasing risk sharing in this intermediate period, and by a sufficiently large amount. Again by differentiating the consumers’ first order conditions with respect to \( \varepsilon \) we obtain the following expression for the price effect at the intermediate date:

\[
\frac{dq_2(1;1)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{q_1^0(2) \left( \beta q_1^0(2)+d\beta \right) u''(h) - \beta q_1^0(2) u''(0) (d q_1^0(2) - \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0}) + 2 q_1^0(2) u''(0) \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0}}{u''(h) - q_1^0(2) u''(h)}.
\]

(31)

Substituting this expression into the sufficient condition for suboptimality we obtained above, (30), we find that this, after rearranging terms, is equivalent to the following:

\[
q_1^0(2) \left( 1-\beta \right) \left[ \beta d - 2 (1-\beta) q_1^0(2) \right] u''(h) - (1-\beta) \beta (q_1^0(2) + d\beta) u''(0) (d q_1^0(2) - \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0})
- (2 q_1^0(2) + d\beta) \left( u'(h) - q_1^0(2) u''(h) \right)
+ \left[ \beta (2-\beta) \left( u'(h) - q_1^0(2) u''(h) \right) + \beta (1-\beta) u'(0) + (1-\beta) \beta (q_1^0(2) + d\beta) u''(0) \right] \frac{dq_1(2)}{d\varepsilon} \bigg|_{\varepsilon=0} > 0
\]

(32)

This condition is stated in terms of endogenous variables which obviously raises the question if there are economies for which the equilibrium values satisfy it. We establish then the following result.
Theorem 6 There are specifications of economies in the environment under consideration that are robust with respect to perturbations in $(h, d, \beta)$ as well as perturbations of preferences for which Condition (32) holds and hence the competitive equilibrium is constrained suboptimal.

To prove the theorem, we show (in the Appendix) that for sufficiently small $\beta$ Condition (32) is satisfied if

$$1 + d\frac{u''(d)}{u'(d)} + \frac{u'(h)}{u'(d)} < 0. \quad (33)$$

As shown in Section 3, when $h(1 - \beta) > d$ an inefficient steady state equilibrium exists with $\frac{u'(h)}{u'(d)} < 1$. It then follows that the inequality $-d\frac{u''(d)}{u'(d)} > 1 + \frac{u'(h)}{u'(d)}$ is satisfied when the absolute risk-aversion is sufficiently high. Therefore Condition (32) holds and the steady state equilibrium is constrained inefficient whenever the agents’ absolute risk aversion is uniformly above $2/d$ and $\beta$ is sufficiently small. It is clear that this is true for an open set of parameters and utility functions.

5.2.1 Logarithmic preferences

While Theorem 6 above is all one can say in general, it is useful to illustrate for a given specification of the agents’ utility function how large the set of parameter values is for which one obtains constrained inefficient equilibria. We consider here the case where $u(c) = \log(c)$.

It can be verified that in this case an explicit solution of (26) for the equilibrium price can be found, given by

$$q_0^1(2) = \beta h^2.$$ 

Since utility is homothetic it is without loss of generality to normalize $d = 1$. Direct computations then show that

$$\left.\frac{dq_2(1; 1)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\beta(1 + h)(1 + \beta h)}{2 + \beta h}$$

and

$$\left.\frac{dq_1(2)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\beta(-4h + \beta^2 h(2 + 3h) + 2\beta(1 + h - 2h^2))}{2(\beta - 2)(2 + \beta h)}.$$ 

According to Equation (30) an improvement is possible if

$$\beta \left.\frac{dq_2(1)}{d\varepsilon}\right|_{\varepsilon=0} + (1 - \beta) \left.\frac{dq_2(1; 1)}{d\varepsilon}\right|_{\varepsilon=0} - \frac{2q_2(1) + d\beta}{(2 - \beta)} > 0$$

Substituting these expressions into (30) we find that, in the case of logarithmic preferences the intervention considered is welfare improving if, and only if

$$2 - \beta(h - 2)h + \beta^2h^2 < 0.$$ 

Figure 2 then shows, in the space $h, \beta$, the region of values of these parameters for which competitive equilibria are constrained inefficient as well as the region where equilibria are
Pareto efficient. We see that the region where constrained inefficiency holds is quite large, while the region where full Pareto efficiency cannot be attained but still the intervention considered is not welfare improving is very small.

6 Appendix: Proofs

6.1 Proof of Theorem 1

We first show that each Arrow Debreu equilibrium allocation with limited pledgeability is also an equilibrium allocation with intermediaries. Given the equilibrium Arrow Debreu prices \( (\rho(\sigma))_{\sigma \in \Sigma} \), set the prices of the tree equal to \( q(s^t) = \frac{1}{\rho(s^t)} \sum_{\sigma \prec s^t} \rho(\sigma) d(\sigma) \) and the prices of the tree-options as

\[
q_{s^t+1}(s^t) = \frac{1}{\rho(s^t)} \rho(s^t+1) \left( q(s^t+1) + d(s^t+1) \right)
\]  

for every \( s^t, s^t+1 \). It is then easy to see that the set of budget feasible consumption levels are the same for the budget set in (IE2) and for the budget set defined by (3) and (4). For any \( h \in H \), given an arbitrary consumption sequence \( (c(\sigma))_{\sigma \in \Sigma} \) that satisfies (IE2), using (34) we get

\[
\rho(s^t) \theta_{s^t}(s^t-1)(q(s^t)+d(s^t)) = \rho(s^t)(c(s^t)-e_h(s^t)) + \rho(s^t) \sum_{s^t+1 \in S} \theta_{s^t+1}(s^t) \frac{\rho(s^t+1)}{\rho(s^t)} \left( q(s^t+1)+d(s^t+1) \right)
\]

for each \( s^t \) with \( t \geq 1 \). Substituting then recursively for the second term on the right hand side we obtain

\[
\rho(s^t) \theta_{s^t}(s^t-1)(q(s^t)+d(s^t)) = \sum_{\sigma \geq s^t} \rho(\sigma)(c(\sigma) - e_h(\sigma)) \geq 0,
\]

that is (4) holds. At the root node \( s_0 \) we have

\[
\theta^h(s^t)(q(s_0)+d(s_0)) = \sum_{\sigma \geq s_0} \rho(\sigma)(c(\sigma) - e_h(\sigma))
\]

equivalent to (3). The reverse implication can be similarly shown.

We show next that an equilibrium allocation with intermediaries is a collateral constrained financial markets equilibrium allocation for a sufficiently rich asset structure \( J \), constructed as follows. In addition to the tree, at each node there are \( S-1 \) financial securities. Security \( j = 1, \ldots, S-1 \) promises a zero payment in all states \( s = 1, \ldots, j \) and a payment equal to 1 in the other states \( j+1, \ldots, S \).

Given any equilibrium with intermediaries, with consumption allocation \( (\bar{c}^h(s^t))_{h \in H} \), prices \( \bar{q}_s(s^t) \) and portfolios \( (\bar{\theta}^h(s^t))_{h \in H} \) of the tree options, for all \( s, s^t \), let

\[
\bar{k} \equiv \sup_{s^t} \left( \sum_s \bar{q}_s(s^t) + d(s^t) \right) < \infty.
\]
Consider the following specification of the collateral requirements of the $J = S - 1$ financial securities:

$$k_{j+1}^i = \frac{1}{k}, \quad k_j^i = 0 \text{ for all } j = 1, \ldots, J$$

$$k_{j-1}^i = 1, \quad k_j^i = 0 \text{ for all } i \neq j - 1, \text{ for all } j = 2, \ldots, J$$

It suffices to show that a collateral constrained financial markets equilibrium exists with the same consumption allocation $(\bar{c}_h(s^t))_{h \in H}$ and tree prices $q(s^t) = \sum_s \bar{q}_s(s^t)$. At this equilibrium, the payoffs of the financial securities are:

$$f_j(s^t) = \begin{cases} 
\frac{q(s^t) + d(s^t)}{k} & \text{if } s_t > j \\
0 & \text{otherwise.} 
\end{cases}$$

and the securities’ prices

$$p_j(s^t) = \frac{1}{k} \sum_{s=j+1}^S \bar{q}_s(s^t), \quad j = 1, \ldots, S - 1.$$

Consider then the following portfolio holdings for each agent $h$, and each node $s^t$: set $\theta^h(s^t) = \bar{\theta}_1^h(s^t)$, $\phi_{-}^h(s^t) = -k\bar{\theta}_1^h(s^t)$, $\phi_{+}^h(s^t) = k\bar{\theta}_2^h(s^t)$, $\phi_{j-1}^h(s^t) = k\bar{\theta}_j^h(s^t)$ and for all other $j = 2, \ldots, J - 1$

$$\phi_{j+}^h(s^t) = k\bar{\theta}_{j+1}^h(s^t), \quad \phi_{j+1}^h(s^t) = -k\bar{\theta}_{j+1}^h(s^t).$$

It is easy to verify that these portfolio holdings, together with the above prices of the tree and the securities, satisfy the collateral constraints, yield the consumption allocation $(\bar{c}_h(s^t))_{h \in H}$ and are so the consumers’ optimal choices.

### 6.2 Proof of Lemma 1

Note first that if there is a positive solution to system (9), it must satisfy $h - q_2(1) \geq l + \delta + q_1(2)$. Suppose this inequality were not satisfied; since we are considering the case where $h - l > \frac{\delta}{1-\beta}$, we would then have $q_2(1) + q_1(2) > \frac{\delta}{1-\beta} - \delta = \frac{\beta \delta}{1-\beta}$. Furthermore,

$$\frac{u^{1'}(l + \delta + q_1(2))}{u^{1'}(h - q_2(1))} < 1 \text{ and } \frac{u^{2'}(l + \delta + q_2(1))}{u^{2'}(h - q_1(2))} < 1.$$

Substituting these inequalities in (9) yields

$$q_2(1) < \frac{\beta \pi_2}{1 - \beta \pi_2} (q_1(2) + \delta) < \frac{\beta \pi_2}{1 - \beta \pi_1} \left( \frac{\beta \pi_1}{1 - \beta \pi_1} (q_2(1) + \delta) + \delta \right)$$

Equivalently, by collecting the terms with $q_2(1)$ on the left hand side and simplifying we obtain

$$q_2(1)(1 - \beta) < \beta \pi_2 \delta.$$
Symmetrically, we can perform the same operation for \( q_1(2) \) to obtain
\[
q_1(2)(1 - \beta) < \beta \pi_1 \delta.
\]
Adding up these two inequalities yields a contradiction to the inequality \( q_2(1) + q_1(2) > \frac{\beta \delta}{1 - \beta} \) above. Therefore a solution to (9) must always satisfy \( h - q_2(1) \geq 1 + \delta + q_1(2) \).

To show that a positive solution to (9) exists, recall the following lemma that follows directly from Brouwer’s fixed point theorem (see e.g. Zeidler (1985), Proposition 2.8).

**Lemma 3** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous function such that
\[
\inf_{\|x\|=r} \sum_{i=1}^{n} x_i f_i(x) \geq 0, \text{ for some } r > 0.
\]
Then \( f \) has at least one zero, i.e. there is a \( x \) with \( \|x\| \leq r \) and \( f(x) = 0 \).

For sufficiently small \( \epsilon > 0 \), define \( g : [-\delta, h - \epsilon]^2 \rightarrow \mathbb{R}^2 \) by
\[
g(x, y) = \begin{cases} 
\frac{1 - \beta \pi_2}{\beta \pi_2} x u^1(h - x) - (y + \delta) u^1(1 + \delta + y) & \\
\frac{1 - \beta \pi_1}{\beta \pi_1} y u^2(h - y) - (x + \delta) u^2(1 + \delta + x) 
\end{cases}
\]
Define \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
\[
f(x, y) = g \left( \max \left[ -\delta, \min [h - \epsilon, x] \right], \max \left[ -\delta, \min [h - \epsilon, y] \right] \right).
\]
We can apply Lemma 3 for \( r = h \), using the sup-norm and obtain the existence of a zero point for \( f \) and then verify that this must also be a solution to \( g(x, y) = 0 \). For \( x = -r \) we obtain
\[
x f_1(x, y) + y f_2(x, y) \geq r(y + \delta) u_1^1(1 + \delta + y) \geq 0.
\]
For \( x = r \) we obtain that \( x f_1(x, y) \) can be made arbitrarily large by choosing \( \epsilon \) appropriately small while \( y f_2(x, y) \) is obviously bounded below since we assume \( t > 0 \). By symmetry, the same is true for \( y \in \{ -r, r \} \) and there must be \((x^*, y^*)\) with \( f(x^*, y^*) = 0 \). It is easy to see that \((x^*, y^*) \in (0, h)^2 \) and therefore also solve \( g(x, y) = 0 \).

### 6.3 Proof of Theorem 2

**Proof.** To construct equilibrium prices, set \( \rho(s_0) = 1 \) and
\[
\rho(s^t) = \rho(s^{t-1}) \beta \pi(s_{t-1}, s_t) \max_{h \in H} \frac{u^h(s_t, \lambda(s^t))}{u^h(s_{t-1}, \lambda(s^{t-1}))}.
\]
Agent \( h \)’s first order conditions for optimal consumption at some node \( s^t \) can be written as follows:
\[
\beta^t \pi(s^t) u^h(c^h(s^t), s_t) - \eta^h \rho(s^t) + \sum_{\sigma: s^t \geq \sigma} \mu^h(\sigma) \rho(s^t) = 0
\]
\[
\mu^h(s^t) \sum_{\sigma \geq s^t} \rho(\sigma) (c^h(\sigma) - e^h(\sigma)) = 0,
\]

34
for multipliers $\eta^h \geq 0$ (associated with the intertemporal budget constraint (3)) and $\mu^h(\sigma) \geq 0$ (associated with the collateral constraint (4) at node $\sigma$). It is standard to show that for summable and positive prices these conditions, together with the budget inequalities (3) and (4) are necessary and sufficient for a maximum (see e.g. Dechert (1982)). But then at each $s^t$ and for all agents $h = 2, ..., H$ we have

$$\frac{u^h_t(c^t(s^t), s^t)}{u^h_t(c^h(s^t), s^t)} = \frac{\eta^h - \sum_{s^t \geq s} \mu^h(\sigma)}{\eta^h - \sum_{s^t \geq s} \mu^h(\sigma)}$$

which is equivalent to the first order conditions of (10) if $1/\lambda^h(\sigma) = \eta^h - \sum_{s^t \geq s} \mu^h(\sigma)$ for all $h, \sigma$. It remains to be shown that the budget inequalities (4) as well as the market clearing conditions are satisfied. The latter is obvious, given (10). Regarding the budget inequalities we need to show that $V^h(s^t, \lambda(s^t)) = 0$ if and only if $\sum_{s^t \geq s} \rho(\sigma)(e^h(\sigma) - e^h(\sigma)) = 0$. Since for any agent $h \in \mathcal{H}$, $\rho(s^{t+1}) = \frac{u^{h_t}(s_{t+1}, \lambda(s^{t+1}))}{u^{h_t}(s_t, \lambda(s^t))}$ whenever $V^h(s_{t+1}, \lambda(s^{t+1})) \neq 0$, this follows from the definition of $V^h$.

### 6.4 Proof of Theorem 3

Given an Arrow-Debreu equilibrium with limited pledgeability we can describe the equilibrium consumption allocation by the associated instantaneous weights $\lambda(s^t)$, which are uniquely determined if we normalize the initial weights $\sum_{h \in \mathcal{H}} \lambda^h(s_0) = 1$ and for all $t > 0$, all $s^t$, require that $\lambda(s^t) \geq \lambda(s^{t-1})$ and $\lambda^h(s^t) = \lambda^h(s^{t-1})$ for at least one agent $h \in \mathcal{H}$.

It is a standard argument to show that for each $l \in \mathbb{R}_{++}^H$, $\sum h^{th} = 1$, there exists an Arrow-Debreu equilibrium with limited pledgeability with $\lambda(s_0) = l$ and some transfers at $t = 0$. To prove the existence of a Markov equilibrium it suffices to show that the equilibrium associated with any given, initial $\lambda(s_0)$ is unique. Suppose to the contrary that there exist two equilibria with instantaneous weights $\lambda_1, \lambda_2$ with $\lambda_1(s_0) = \lambda_2(s_0)$ but $\lambda_1(s^t) \neq \lambda_2(s^t)$ for some $s^t$. Define for each $s^t$ and all $h \in \mathcal{H}$, $\Delta^h(s^t) = \min(\lambda^1_1(s^t), \lambda^2_1(s^t))$. Since both $\lambda_1(s^t)$ and $\lambda_2(s^t)$ describe equilibria we must have $\Delta(s^t) \neq \lambda_1(s^t)$ and $\Delta(s^t) \neq \lambda_2(s^t)$ for some $s^t$. Define recursively

$$v^h(s^t) = u^{h_t}(s_t, \Delta)(C^{h_t}(s_t, \lambda) - e^h(s^t)) + \beta \sum s^t \pi(s_t, s^t) v^h(s^{t+1}).$$

By Lemma 4 below we have, for each $s^t$, $u^{h_t}(s_t, \lambda(s^t))(C^{h_t}(s_t, \Delta(s^t)) - e^h(s_t)) \geq \min \left[ u^{h_t}(s_t, \lambda_1(s^t))(C^{h_t}(s_t, \lambda_1(s^t)) - e^h(s_t)), u^{h_t}(s_t, \lambda_2(s^t))(C^{h_t}(s_t, \lambda_2(s^t)) - e^h(s_t)) \right]$ and therefore $v^h(s_0) \geq 0$. By the first order conditions of the Negishii maximization problem the terms $\lambda^h u^h(s, \lambda)$ are identical across all agents $h$, for all $s$ and $\lambda$, hence the Arrow-Debreu prices in the two equilibria are given by $\rho_i(s^t) = \beta \pi(s^t) \lambda^i_1(s^t) u^{i_t}(s_t, \lambda_i(s^t))$ for

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16Kubler and Schmedders (2003) show existence for all initial levels of tree-holdings, the same technique can then be applied to all initial Negishii-weights.
i = 1, 2. Since the price of the tree is finite, prices are summable, each \((\beta^t \pi(s') \lambda^h(s'))\) is also summable and so is \((\beta^t \pi(s') \lambda^h(s'))\). Define \(\rho(s') = \beta^t \pi(s') \lambda^h(s') u_s(t, \Lambda(s'))\) for all \(s'\).

Lemma 4 also implies \(\lambda^h(s') u_s(t, \Lambda(s')) \leq \min [\lambda^h_1(s') u_s(t, \Lambda(s'))], \lambda^h_2(s') u_s(t, \Lambda(s'))\] for all \(s'\), with the inequality holding strict for some \(s'\). Therefore the allocation \(g(s') = C(s_t, \Lambda(s'))\) would have to satisfy

\[
\sum_{s'} \rho(s') \left( \sum_{h \in H} (e^h(s') - e^h(s_t)) - d(s_t) \right) > 0.
\]

Since \(\rho\) is summable this contradicts feasibility and hence the equilibrium must be unique.

**Lemma 4** Suppose that for all \(s, h, u^h(c, s)\) satisfies the property that \(c u^h(c, s)\) is (weakly) increasing in \(c\). For any \(\lambda_1, \lambda_2 \in \mathbb{R}^H\), let \(\lambda^h = \min[\lambda^h_1, \lambda^h_2], h = 1, ..., H\); if \(\lambda^h \neq \lambda_1 \) and \(\lambda^h \neq \lambda_2\), for all \(h\) we have

\[
u^h_1(s, \lambda^h) (C^h(s, \lambda) - e^h(s)) > \min \left[ \nu^h_1(s, \lambda^h) (C^h(s, \lambda^h) - e^h(s)), \nu^h_2(s, \lambda^h) (C^h(s, \lambda^h) - e^h(s)) \right]
\]

as well as

\[
\lambda^h u^h(s, \lambda) < \min \left[ \lambda^h_1 u^h(s, \lambda^h_1), \lambda^h_2 u^h(s, \lambda^h_2) \right]
\]

**Proof.** Assume without loss of generality that \(\lambda^h = \lambda^h_1 \leq \lambda^h_2\). Since \(\lambda = \lambda_1\) we must have \(C^h(s, \lambda) > C^h(s, \lambda_1)\) and so (35) follows from the assumption made on \(u^h(.)\). Concavity of \(u^h(.)\) implies that \(\lambda^h u^h(s, \lambda) < \lambda^h_1 u^h(s, \lambda_1)\). To prove that \(\lambda^h u^h(s, \lambda) \leq \lambda^h_2 u^h(s, \lambda_2)\) and therefore (36) holds, define \(\bar{\lambda}\) by \(\bar{\lambda}^h = \bar{\lambda}^h_1 \leq \bar{\lambda}^h_2\) and \(\bar{\lambda}^h_i = \lambda^h_2\) for all \(i \neq h\). Since \(\lambda^h u^h(s, \lambda) = \bar{\lambda}^h u^h(s, \lambda)\), we must have \(\lambda^h u^h(s, \lambda) \leq \lambda^h u^h(s, \lambda_2)\). Furthermore, we have \(\lambda^h u^h(s, \lambda) \leq \lambda^h u^h(s, \lambda)\). Also one of the last two inequalities must hold strict, so that (36) follows. \(\square\)

### 6.5 Proof of Lemma 2

To show the existence of a solution of (15)-(17), we can substitute out all \(V^1(s, \bar{\lambda}(\bar{s}))\) and \(V^2(s, \bar{\lambda}(\bar{s}))\) as well as all \(V^1(s, \bar{\lambda}(\bar{s}))\) and \(V^2(s, \bar{\lambda}(\bar{s}))\) for \(s \neq \bar{s}\). We obtain a function \(f : (0, 1)^{2S} \rightarrow \mathbb{R}^{2S}\), where each \(f_i, i = 1, ..., S\) is the weighted sum of terms of the form

\[
u^1(s, \bar{\lambda}(\bar{s})) (C^1(s, \bar{\lambda}(\bar{s})) - e^1(s)) \text{ and } u^1(s, \bar{\lambda}(\bar{s})) (C^1(s, \bar{\lambda}(\bar{s})) - e^1(s)),
\]

where the weights on the terms involving \(\bar{\lambda}(s)\) are positive (bounded away from zero) if and only if there is an \(s'\) with \(\bar{\lambda}(s') > \bar{\lambda}(s)\) (recall that \(\pi(s, s') > 0\) for all \(s, s')\). Similarly each \(f_i\) with \(i = S + 1, ..., 2S\) is a weighted sum of terms

\[
u^2(s, \bar{\lambda}(\bar{s})) (C^2(s, \bar{\lambda}(\bar{s})) - e^2(s)) \text{ and } u^2(s, \bar{\lambda}(\bar{s})) (C^2(s, \bar{\lambda}(\bar{s})) - e^2(s)),
\]
where the weights on the terms involving \( \lambda(s) \) are positive if and only if there is an \( s' \) with \( \lambda(s') < \lambda(s) \). We obtain that \( f(\lambda(1), \lambda(1), \ldots, \lambda(S), \lambda(S)) = 0 \) precisely when there exists a solution to (15) and (16) with

\[
V^1(s, \lambda(s)) = V^2(s, \lambda(s)) = 0 \text{ for all } s \in S.
\]

To prove the lemma it therefore suffices to show that for sufficiently small \( \epsilon > 0 \), there exist \( x \in [\epsilon, 1 - \epsilon]^{2S} \) with \( f(x) = 0 \).

This result follows directly by applying Lemma 3 to a slight modification of the function \( f(.) \). For \( x \in [\epsilon, 1 - \epsilon]^{2S} \), set \( g_i(x) = f_i(x) \) for \( i = 1, \ldots, S \) and \( g_i(x) = -f_i(x) \) for \( i = S + 1, \ldots, 2S \). Extend then the function \( g \) to the whole domain \( \mathbb{R}^{2S} \) by setting it continuous and constant outside of \([\epsilon, 1 - \epsilon]^{2S}\). All one needs to prove is the appropriate boundary behavior. Clearly as some \( \lambda(s) \) is sufficiently large or some \( \lambda(s) \) is sufficiently small, we have that \( \sum_i x_i g_i(x) < 0 \) since each \( f_i(x) \) is bounded above. The key is to show that if \( \lambda(s) \) is sufficiently small, or if \( \lambda(s) \) is sufficiently large, we also have that some \( |g_i(x)| \) becomes arbitrarily large. To show this note that in (38) the terms involving \( \lambda(s) \) have positive (and bounded away from zero) weight whenever there is a \( s' \) with \( \lambda(s') < \lambda(s) \). If this is the case clearly some \( f_i(x), i = 1, \ldots, S \) can be made arbitrarily small; if it is not the case, some \( \lambda(s') \) becomes arbitrarily close to 1 and we are in the case above. The argument for \( \lambda(s) \) is analogous.

**6.6 Further details on the argument for constrained inefficiency**

**Derivation of Equation (26).** At \( t = 1 \) in state 1 the price \( q_2(1; 1) \) of the tree option paying in state 2 is determined by agent 1’s first order condition, since agent 2 is constrained in that state in his holdings of that asset. We have so

\[
q_2(1; 1)u'(q - q_1(1; 1)) - q_2(1; 1)(1 - \epsilon) = \frac{\beta}{2} (q_1(2) + q_2(2) + \delta)u'(\delta(1 - \epsilon) + q_1(2)(1 - 2\epsilon)).
\]  

(39)

For \( t \geq 2 \) agent 1’s first order conditions with respect to the tree option paying in state 2 still determine its price in state 1 since agent 2 is constrained in that state. On the other hand, in state 2 the consumption of both agents is the same as in the subsequent date in state 2, hence both agents are not constrained in their holdings of the tree options paying in state 2 and its price is determined by the first order conditions of any of them (say again agent 1).

\[
q_2(1)u'(q + \epsilon\delta - q_2(1)(1 - 2\epsilon)) = \frac{\beta}{2} (q_1(2) + q_2(2) + \delta)u'(\delta(1 - \epsilon) + q_1(2)(1 - 2\epsilon))
\]  

(40)

\[
q_2(2)u'(\delta(1 - \epsilon) + q_2(2)(1 - 2\epsilon)) = \frac{\beta}{2} (q_1(2) + q_2(2) + \delta)u'(\delta(1 - \epsilon) + q_1(2)(1 - 2\epsilon))
\]  

(41)

From (41) we obtain for \( t > 1 \) that

\[
q_1(1) = q_2(2) = \frac{\beta(q_1(2) + \delta)}{2 - \beta}
\]  

(42)

37
and therefore

\[ q_1(2) + q_2(2) + \delta = \frac{2(q_1(2) + \delta)}{2 - \beta}. \]

Substituting this expression into equation (40) we obtain (26).

**Derivation of Equation (31).** The expression for the price change in (31) is obtained by differentiating (39) with respect to \( \varepsilon \), evaluated at \( \varepsilon = 0 \), when \( q_2(1; 1), q_1(2) \) and \( q_1(1; 1), q_2(2) \) are at their steady state values before the intervention, given respectively by \( q_1^0(2) \) for the first two and by \( \beta (q_1(2) + \delta) \) for the last two. Noting that \( \frac{d q_1(1; 1)}{d \varepsilon} \bigg|_{\varepsilon=0} = 0 \), since the price \( q_1(1; 1) \) also changes with \( \epsilon \) but the expression is evaluated at \( \epsilon = 0 \), we get (31).

**Derivation of Condition (33).** From equation (26) we find that \( q_1^0(2) \) can be written in term of \( u'(\delta) \) and \( u'(h) \),

\[ q_1^0(2) = \frac{\beta}{2 - \beta} \frac{u'(\delta)}{u'(h)} - \frac{\beta}{2 - \beta} \frac{u'(\delta)}{u'(h)} (43) \]

Defining \( \bar{u}'(\delta) := \frac{u'(\delta)}{u'(h)} \), \( \bar{u}''(\delta) := \frac{u''(\delta)}{u''(h)} \), and \( \bar{u}''(h) := \frac{u''(h)}{u''(h)} \) we obtain that Condition (32) is equivalent to the condition \( \frac{A}{B} > 0 \) where

\[ A = 4\bar{u}'(\delta) \left[ 1 + \delta \bar{u}''(\delta) + \bar{u}'(\delta) \right] - 2\beta \left[ 2 + (4 + 3\delta \bar{u}''(\delta)) \bar{u}'(\delta) + 2\bar{u}'(\delta)^2 + \delta \bar{u}''(\delta) \right] + \beta^2 \left[ \frac{1}{\bar{u}'(\delta)} + (3 + 2\delta \bar{u}''(\delta)) \bar{u}'(\delta) + \bar{u}'(\delta)^2 + \delta \bar{u}''(\delta) + (3 + \delta \bar{u}''(\delta)) \right] \]

and

\[ B = [-2\bar{u}'(\delta)^2 + \beta (\bar{u}'(\delta) + \bar{u}'(\delta)^2 + \delta \bar{u}''(\delta))] \left[ 4\bar{u}'(\delta)^2 + \beta^2 (1 + (2 + \delta \bar{u}''(\delta)) \bar{u}'(\delta) + \bar{u}'(\delta)^2 + \delta \bar{u}''(\delta)) \right] - 2\beta ((2 + \delta \bar{u}''(\delta)) \bar{u}'(\delta) + 2\bar{u}'(\delta)^2 + \delta \bar{u}''(\delta)) \]

It can then be easily seen that, since all marginal utilities are evaluated at positive numbers, that remain bounded away from zero as \( \beta \to 0 \), for sufficiently small \( \beta \) we have \( \frac{A}{B} > 0 \) if \( 1 + \delta \bar{u}''(\delta) + \bar{u}'(\delta) < 0 \), or equivalently

\[ 1 + \delta \frac{u''(\delta)}{u'(\delta)} + \frac{u'(h)}{u'(\delta)} < 0. \]

**References**


Figure 1: Finite support equilibrium
Figure 2: Constrained inefficient region