Constrained Inefficiency and Optimal Taxation with Uninsurable Risks*

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Abstract
When individuals' labor and capital income are subject to uninsurable idiosyncratic risks, should capital and labor be taxed, and if so how? In a two period general equilibrium model with production, we derive a decomposition formula of the welfare effects of these taxes into insurance and distribution effects. This allows us to determine how the sign of the optimal taxes on capital and labor depend on the nature of the shocks, the degree of heterogeneity among consumers' income as well as on the way in which the tax revenue is used to provide lump sum transfers to consumers. When shocks affect primarily labor income and heterogeneity is small, the optimal tax on capital is positive. However in other cases a negative tax on capital is welfare improving. (JEL codes: D52, H21. Keywords: optimal linear taxes, incomplete markets, constrained efficiency)

1 Introduction
The main objective of this paper is to investigate the role and optimal form of taxation of investment and labor income in a two period production economy with uninsurable background risk. More precisely, we investigate whether the introduction of linear, distortionary taxes or subsidies on labor income and/or on the returns from savings are welfare improving and what is then the optimal sign of such taxes. This amounts to studying the Ramsey problem in a general equilibrium set-up. We depart however from most of the literature on the subject for the fact that we consider an environment with no public expenditure, where there is no need to raise tax revenue. Still, optimal taxes are typically nonzero as we will show. The reason is that even distortionary taxes can improve the allocation of risk in the face of incomplete markets. The issue then arises of which production factor should be taxed, and which economic properties determine the signs of the optimal taxes.

A possible answer to this question may come from the following consequence of the agents’ precautionary motive for saving: under uninsurable risk, this motive implies that savings and hence capital accumulation will be higher compared to the situation where markets are complete. This point was made in an influential paper by Aiyagari (1995, p. 1160) and in fact various papers

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1See Chari and Kehoe (1999) for a survey.
thereafter suggested, implicitly and explicitly, that with incomplete markets, the precautionary saving motive leads to over-accumulation of capital and hence that a positive tax on capital is welfare improving.\(^2\)

We argue however that this implication is unwarranted: the comparison between the level of capital accumulation with and without complete markets has no clear welfare implication. If there were a policy tool which could implement the complete market allocation, there would be little doubt for the policy maker to adopt such a policy as far as attaining efficiency is concerned. However, since a tax and subsidy scheme of the kind mentioned above will not allow to complete the markets, this comparison tells little about the welfare effects of taxation, not to mention whether or not capital should be taxed.

To properly assess whether or not positive taxes on capital are welfare improving when markets are incomplete, one should rather compare the competitive equilibria with and without taxes, keeping the other parts of the market structure, and in particular the set of available financial assets, fixed. This “second best” exercise is what we do formally in this paper. We consider explicitly market equilibria with taxation, for various tax-subsidy schemes. We say that capital should be taxed (resp. subsidized) if there is an equilibrium with a positive (resp. negative) tax on capital where consumers’ welfare is higher than in an equilibrium without tax. The same applies for labor.

The reader may still wonder if the optimal capital tax should ever be negative in the sense above when the equilibrium stock of capital is higher than when markets are complete. We show that indeed subsidizing capital may be welfare improving in such a case. This finding does not rely on the presence of upward sloping demand curves, so that subsidizing capital further increases its level but nevertheless raises consumers’ welfare. To give some intuition for this, let us first describe the model more explicitly to outline our results.

We consider a two period economy with production, where the savings of each consumer can be invested to obtain capital, which is then used as input in the production process the next period. In addition, the consumer has to choose how much to work in the second period, and the productivity of his work is subject to idiosyncratic shocks. The amount of capital obtained per unit invested by a consumer may also be subject to idiosyncratic shocks. This is all the uncertainty in the model, idiosyncratic shocks are independent across consumers and there is a continuum of consumers so that there is no aggregate uncertainty. Consumers may differ in terms of their initial income as well as of their preferences. Capital and labor are exchanged between consumers and firms in competitive markets. Since the consumers’ investment is the only instrument allowing them to transfer income over time, markets are clearly incomplete. Linear, uniform taxes on the wage income as well as on the investment income may be introduced and the net revenue from these taxes is redistributed to consumers via lump sum transfers. Such taxes and transfers affect individual savings and labor supply decisions and thus induce a change in equilibrium prices, and of course in consumers’ welfare.

With incomplete markets these price changes affect the risk allocation among agents, a pecuniary externality which was first noticed by Hart (1975) and Stiglitz (1982). In our setup, the pecuniary externality consists of two effects. First, a price change has an insurance effect: if the price of a risky factor of production goes down, the risk in the agent’s future income is reduced, thus the consequences of the missing markets are partially offset. Secondly, a price change also has an indirect distribution effect as it does with complete markets: since agents are endowed differently in the factors of production, a change in the relative prices of factors will induce transfers of income across different types of consumers. In addition, depending on the way in which the revenue of the

tax is redistributed to consumers, there may be some direct distribution effect as well: for instance, if the lump sum transfer is constant and common to every consumer, it will directly provide some income smoothing across consumers.

Even in this relatively simple environment to determine the actual level of the optimal tax rates is not an easy task. We shall therefore primarily focus our analysis on the effects of introducing taxes at an infinitesimal level at a competitive equilibrium with no taxes. The first order effects of such taxes can be obtained by differentiating the equilibrium system, whenever the equilibrium with no taxes is regular. We will decompose them into insurance and distribution effects and investigate their properties. Building on this decomposition result, we are then able to identify sets of conditions (concerning both the characteristics of the economy and of the tax schemes considered) under which a welfare improvement can be obtained with a positive, or a negative, tax on capital, and similarly for labor.

In Section 3, as a preliminary step, we consider a situation where consumers’ savings or labor supply can be directly modified, which we shall refer to as the constrained optimality exercise. This problem was also investigated by Davila et al (2005) in a similar setup, but with an exogenous labor supply and with idiosyncratic shocks only affecting labor productivity. We find that the insurance effect operates in favor of a decrease in consumers’ investment, or alternatively of an increase in labor supply, when the shocks affect primarily labor rather than capital (and vice versa in the opposite case). With significant heterogeneity in the pattern of consumers’ initial income (and possibly preferences), the distribution effect also becomes important. This effect has a different sign across consumers: while the relatively poor consumers benefit even more from a decrease in the investment level, the relatively rich ones might actually loose from it. As a consequence, when heterogeneity among the different types of consumers is large, a change (in investment or labor supply) improving the utility of all types of consumers cannot be found. Therefore we investigate which changes improve social welfare evaluated ex ante, ‘under the veil of ignorance’, that is before a type is assigned to any consumer. A redistribution among different types of consumers may then be justified as a way to provide agents some insurance against ‘bad’ realizations of their type.

We find that the distribution effect works for ex ante welfare in the opposite direction as the insurance effect. Hence, considering again the case where shocks primarily affect labor, if investment is reduced ex ante welfare increases (and we then say there is over investment in equilibrium) when the heterogeneity among consumers of different types is small, but it worsens when the heterogeneity is large. Opposite conclusions hold for labor supply changes. We emphasize that these findings are independent of the level of the equilibrium interest rate and of the presence of a precautionary motive. Therefore, to determine whether or not there is over investment in equilibrium, a primary role is played by the degree of heterogeneity among consumers and by the nature of the shocks.

We turn then to the analysis of optimal taxes in Section 4. With taxes, savings and labor supply can only be controlled indirectly and the welfare effects of taxes depend on the way they affect savings and labor supply.

When the tax revenue is rebated to consumers without inducing any reallocation of income across consumers or realization of the idiosyncratic shocks, the effects of taxes are analogous to those of directly controlling investment or labor supply (Section 4.1). We then find that the optimal tax on capital is positive (and the optimal tax on labor negative) exactly when there is over investment in the constrained optimality exercise. Therefore when the shocks affect primarily labor and the degree of heterogeneity among consumers’ income is sufficiently limited, capital should be taxed. The reverse conclusion holds instead when the heterogeneity is large (or the shocks affect primarily the returns on savings).
In Sections 4.2 and 4.3 we turn our attention to the case where the tax revenue is no longer redistributed to each consumer exactly in proportion to the consumer’s tax payments in each individual state, so that the tax scheme also provides some insurance and/or some income redistribution among consumers. In this situation the basic trade off is as follows: the provision of insurance strengthens the case for a positive tax, especially for the factor whose income is more affected by the shocks. In contrast, the provision of redistribution tends to strengthen the case for taxing capital and weakens that for taxing labor, since it is typically the case that the main source of income is capital for wealthy consumers and labor for poor consumers and transfers from rich to poor are beneficial from an ex ante perspective. Thus the sign of the optimal tax depends on both the sign and the relative importance of these two effects.

We also consider (in Section 4.4) the case where lump sum transfers are not available, so that the revenue of the tax on one factor is redistributed to consumers via a subsidy on the other factor. Surprisingly enough, we obtain that it is optimal to tax capital whenever there is under investment in equilibrium. This is exactly the opposite of what we found when lump sum transfers are possible, without redistribution (in Section 4.1).

The analyses thus far are local, in the neighborhood of a competitive equilibrium with no taxes. The characterization of the optimal level of the tax rates requires a global analysis of the equilibria with taxes and it is then difficult to obtain general results for this. In Section 5 we consider a numerical example of an economy exhibiting standard properties, for which the optimal tax rates are derived for the different tax schemes considered. The numerical results also show that the sign of the optimal tax rates are typically in accord with our findings from the local analysis and illustrate once again how the level of the optimal tax on capital and labor depends on the degree of heterogeneity among consumers.

The two period framework of the analysis allows us to obtain a clear decomposition of the effects of taxes in incomplete market economies and to determine their properties. In such environment, however, we cannot address important issues such as the intertemporal allocation of the tax burden, which has been one of the major issues in the literature, starting with Judd (1985) and Chamley (1986) in a complete market setting and with Aiyagari (1995) when markets are incomplete. Still, we claim that the findings of our present analysis allows to gain some insights on the properties of optimal taxes in dynamic, infinite horizon economies. Building on this work in fact, in a companion paper Gottardi, Kajii and Nakajima (2011) study the solutions of the dynamic Ramsey problem of finding optimal public debt and linear taxes on capital and labor in an infinite horizon model with incomplete markets.

2 The Economy

We consider a two period competitive market economy as follows. The economic agents consist of one representative firm and I types of consumers, with a continuum of consumers of size one for each type.

The firm has a constant returns to scale technology described by a smooth homogeneous concave production function $F(K, L)$, where the output is the amount produced of the single consumption good, per capita, $K$ is the amount of capital input, also per capita, and $L$ is the amount of labor input per capita, both measured in efficiency units (as made clearer in what follows). The firm maximizes profits taking prices as given: writing $r$ for the cost of capital per efficiency unit and $w$ for the wage per efficiency unit, $K$ and $L$ will be chosen so that $F_K(K, L) = r$ and $F_L(K, L) = w$. The firm operates in the second period, when both the production activity and the purchases of
inputs take place, although other interpretations are possible.

Consumers of the same type are ex ante identical, i.e., have the same preferences and endowment profile and make the same choices in the first period. Each consumer of type $i$ is endowed with $e_i > 0$ units of consumption good in the first period, which may be consumed or invested. If invested, it will yield some amount of the capital good next period (which may also be interpreted as human capital), to be sold to the firm at price $r$. Denote by $k_i$ the amount invested by type $i$, thus $e_i - k_i$ is the consumption in the first period. In the second period, any type $i$ consumer is endowed with $H_i$ units of labor hour ($H_i > 0$) which can be supplied in the market.

Each consumer is subject to idiosyncratic risk. For each $i$, denote by $(\Theta_i, P_i)$ the probability space which describes the shock affecting type $i$ consumers. We assume that the shock is independently and identically distributed across the consumers of type $i$, and independently distributed across different types.

The idiosyncratic shock affects both the return of the consumers’ investment and the efficiency of the consumers’ labor. In state $\theta_i \in \Theta_i$, an investment of $k_i$ units in the first period by a type $i$ consumer yields

$$K_{i,\theta_i} := \rho^K_i(\theta_i)k_i$$

in efficiency units of capital in the second period, where $\rho^K_i$ is a random variable on the state space $(\Theta_i, P_i)$. We further assume that the i.i.d. assumption of the shocks implies that the aggregate supply of capital $K_i$ from type $i$ consumers in efficiency units is equal to $k_i$ times the expected value of the returns obtained from the consumers’ investment, $\gamma_i := E[\rho^K_i(\theta_i)]$, that is $K_i = \gamma_i k_i$. By definition the aggregate per-capita supply of capital is given by $K = \frac{1}{n} \sum_i \gamma_i k_i$.

The level of the labor supply is chosen after $\theta_i \in \Theta_i$ is realized: writing $h^{\theta_i}_i$ for the labor hours supplied after the consumer observed $\theta_i$, the labor supply in efficiency units $L^{\theta_i}_i$ is defined by

$$L^{\theta_i}_i := \rho^L_i(\theta_i)h^{\theta_i}_i,$$

where $\rho^L_i$ is another random variable on $(\Theta_i, P_i)$. We normalize units so that $E[\rho^L_i(\theta_i)] = 1$ for every $i$. Again we assume that the aggregate supply of labor of type $i$ consumers in efficiency units, $L_i$, is equal to the expected level of the labor supply. That is, if we write $L_i$ for the total supply of labor in efficiency units by the consumers of type $i$, $L_i := E[L^{\theta_i}_i]$ holds, and then by definition the aggregate per-capita supply labor is given by $L = \frac{1}{n} \sum_i E[L^{\theta_i}_i]$. In the special case of inelastically supplied labor, $h^{\theta_i}_i = \bar{H}_i$ at every $\theta_i$. In such a case, $L^{\theta_i}_i = \rho^L_i(\theta_i)\bar{H}_i$, and $L_i = \bar{H}_i$.

The structure of the uncertainty thus allows both for idiosyncratic labor income risk, as in Aiyagari (1994), and idiosyncratic capital income risk, as in Angeletos (2007). Allowing for both capital and labor risks generates some symmetry and allows us to identify the role played by each type of risk in the comparative statics and welfare analysis. The special case where there is no shock to capital income, i.e. $\rho^K_i$ is constant and only labor efficiency is subject to idiosyncratic shocks, constitutes an important benchmark and we will refer to it as the benchmark case. It will be the main focus of the parts of the analysis where the sign of the optimal tax rates are determined.

To ensure that the model is well defined, we assume throughout our analysis that both the individual labor endowment and the gross return on savings are always positive: that is, $\rho^K_i(\theta_i) > 0$ and $\rho^L_i(\theta_i) > 0$ occur with probability one. To preserve the uninsurable nature of the consumers’ idiosyncratic risks, we shall also assume that the two random variables $\rho^L_i$ and $\rho^K_i$ are comonotonic, i.e., $(\rho^L_i(\theta_i) - \rho^L_i(\theta'_i))(\rho^K_i(\theta_i) - \rho^K_i(\theta'_i)) \geq 0$ for any pair of states $\theta_i$ and $\theta'_i$. That is, if the labor

\[3\text{Since the shocks are independent, the meaning of the expectation will be clear and so we shall omit the reference to the underlying measure } P_i.\]
endowment of a type \( i \) household is relatively large, the productivity of capital tends to be high as well. We shall therefore use the convention that the household is (relatively) rich at state \( \theta \), if the corresponding \( \rho^L_\theta (\theta_i) \) is (relatively) large. Notice that this assumption holds automatically in the benchmark case.

A type \( i \) consumer’s risk preferences are represented by a time additively separable utility function: the first period utility is given by a function \( u_i \) of the consumption of the good and leisure. So when a type \( i \) individual chooses to invest \( k_i \) and supply \( L^{\theta_i}_i = \rho^L_i (\theta_i) \bar{H} \), efficiency units of labor at \( \theta_i \), he consumes \( e_i - k_i \) units of the good in the first period, and \( wL^{\theta_i}_i + rK^{\theta_i}_i \) units of the good and \( \bar{H}_i - h^{\theta_i}_i \) units of leisure in the second period in state \( \theta_i \), where \( K^{\theta_i}_i = \rho^K_i (\theta_i) k_i \). Thus his choice problem is given as follows:

\[
\max_{k_i, \left( h^{\theta_i}_i \right)_{i \in \theta_i}} \nu_i (e_i - k_i) + \mathbf{E} \left[ u_i \left( wL^{\theta_i}_i + rK^{\theta_i}_i, \bar{H}_i - h^{\theta_i}_i \right) \right],
\]

(3)

where \( K^{\theta_i}_i \) and \( L^{\theta_i}_i \) are defined and understood as functions of \( k_i \) and \( h^{\theta_i}_i \) as in (1) and (2). We assume that both \( v_i \) and \( u_i \) are smooth and concave, strictly increasing in the consumption good and non-decreasing in leisure. We also assume that the random variables are well behaved so that the first order approach is valid: i.e., we assume that the following first order condition completely characterizes the solution to the consumer’s choice problem:

\[
-v'_i (e_i - k_i) + \mathbf{E} \left[ u_{ic} \left( wL^{\theta_i}_i + rK^{\theta_i}_i, \bar{H}_i - h^{\theta_i}_i \right) \cdot r \rho^K_i (\theta_i) \right] = 0.
\]

(4)

\[
u_{ic} \left( wL^{\theta_i}_i + rK^{\theta_i}_i, \bar{H}_i - h^{\theta_i}_i \right) \rho^L_i (\theta_i) w - u_{il} \left( wL^{\theta_i}_i + rK^{\theta_i}_i, \bar{H}_i - h^{\theta_i}_i \right) = 0, \text{ at every } \theta_i,
\]

(5)

where \( u_{ic} \) and \( u_{il} \) stand for the partial derivatives with respect to consumption and leisure, respectively. This assumption is satisfied, for instance, if each state space is finite. Similar convention will be used throughout the paper, e.g., \( u_{ic} \) stands for the second derivative with respect to consumption, and \( u_{il} \) stands for the cross derivative. Furthermore, for the special case where \( u_{il} \) is always equal to zero, labor is inelastically supplied and condition (5) is replaced with

\[
\rho^L_i (\theta_i) \bar{H}_i - L^{\theta_i}_i = 0 \text{ at every } \theta_i.
\]

(6)

Note that, since all individuals of the same type solve the same problem and such problem is convex, their optimal decisions are also the same.

It can be readily verified that the consumption good markets clear when all the factor markets clear. So in this economy a competitive equilibrium occurs when the firm’s profit maximization condition is satisfied at a level of the aggregate input variables that is equal to the one derived from the consumers’ maximization problems. Formally,

**Definition 1** A collection \( \left( \hat{w}, \hat{r}, (\hat{\bar{H}}_i, (\hat{h}^{\theta_i}_i)_{\theta_i \in \Theta_i}) \right)_{i=1}^l \) constitutes a competitive equilibrium if, for each \( i \), \( (\hat{\bar{H}}_i, (\hat{h}^{\theta_i}_i)_{\theta_i \in \Theta_i}) \) is a solution to (3), and the profit maximization conditions, \( F_K \left( \hat{K}, \hat{L} \right) = \hat{r} \) and \( F_L \left( \hat{K}, \hat{L} \right) = \hat{w} \), hold for \( \hat{K} = \frac{1}{2} \sum_{i=1}^l \gamma_i \hat{K}_i \) and \( \hat{L} = \frac{1}{2} \sum_{i=1}^l \mathbf{E} \left[ \hat{L}^{\theta_i}_i \right] \).

By construction, equilibrium labor hours \( (\hat{h}^{\theta_i}_i)_{\theta_i \in \Theta_i} \) for every type \( i \) consumer are a function of the realization of \( \theta_i \), so are the capital in efficiency units, the second period consumption, and the labor supply in efficiency units. Let, in particular, \( e_i^{\theta_i} := \hat{w} \bar{L}^{\theta_i}_i + \hat{r} \hat{K}^{\theta_i}_i \) and \( h_i^{\theta_i} := \hat{H}_i - h^{\theta_i}_i \). The comparative statics properties of the equilibrium and the sign of the optimal taxes depend on how
these variables vary with respect to \( \theta_i \), \( i = 1, ..., I \). We show in what follows that in 'normal cases', that include the situation where consumption and leisure are normal goods and distributions of the shocks as in the benchmark case, the following property holds.

**Definition 2** A competitive equilibrium \( (\tilde{w}, \tilde{r}, (\tilde{k}_i, (\tilde{h}_i^{\theta_i}))_{\theta_i \in \Theta_i}) \) is said to exhibit a **standard response to shocks** (in short, is **standard to shocks**) if, for every \( i \):

i) \( u_{ic} \left( \hat{\epsilon}_i^{\theta_i}, \hat{p}_i^{\theta_i} \right) \) is decreasing in \( \rho_i^L (\theta_i) \); i.e., \( u_{ic} \left( \hat{\epsilon}_i^{\theta_i}, \hat{p}_i^{\theta_i} \right) < u_{ic} \left( \epsilon_i^{\theta_i}, p_i^{\theta_i} \right) \) holds whenever \( \rho_i^L (\theta_i) < \rho_i^L (\theta_i) \).

ii) \( K_i^{\theta_i} \) is non-decreasing and \( \hat{L}_i^{\theta_i} \) is increasing in \( \rho_i^L (\theta_i) \).

Regarding condition i), intuitively speaking, a high realization of \( \rho_i^L (\theta_i) \) implies that the consumer is rich ex post, and so consumption should be relatively high and hence the marginal utility from consumption relatively low. Also, the amount of labor in efficiency units should be relatively high. The following result can then be proved by applying usual consumer theory (a proof is provided for completeness in the Appendix).

**Lemma 1** Assume that \( u_i \) is strictly concave \( (u_{icc}u_{iitl} - (u_{ic})^2 > 0 \) everywhere) and consumption is a normal good \( (u_{icc}u_{iitl} - u_{ic}u_{ic} < 0 \) everywhere). Then in any competitive equilibrium \( (\tilde{w}, \tilde{r}, (\tilde{k}_i, (\tilde{h}_i^{\theta_i}))_{\theta_i \in \Theta_i}) \), condition i) of Definition 2 holds.

Condition ii) is slightly trickier: labor supply \( \hat{L}_i^{\theta_i} = \rho_i^L (\theta_i) \hat{h}_i^{\theta_i} \) is obviously increasing in \( \rho_i^L (\theta_i) \) when \( u_i \) is constant in leisure (and hence the supply of labor hours is inelastic). But when \( u_i \) is increasing in leisure, it is not clear-cut whether or not \( \hat{L}_i^{\theta_i} \) is still increasing in \( \rho_i^L (\theta_i) \): a higher \( \rho_i^L (\theta_i) \) means a higher effective wage which induces more labor, but it also generates a higher income from capital which induces more leisure. So \( \hat{L}_i^{\theta_i} \) should be increasing, roughly speaking, when the income effect from the higher revenue from the capital investment is not excessively large. A formal sufficient condition is stated in the following (also proved in the Appendix):

**Lemma 2** Assume that \( \rho_i^L (\theta_i) \) is constant, and that \( u_i \) is strictly concave and leisure is a normal good \( (u_{itl}u_{ic} - u_{ictl}u_{itl} < 0 \) everywhere). Then in any competitive equilibrium \( (\tilde{w}, \tilde{r}, (\tilde{k}_i, (\tilde{h}_i^{\theta_i}))_{\theta_i \in \Theta_i}) \), condition ii) of Definition 2 holds.

An alternative sufficient condition is the case of inelastic labor hour supply. Indeed in such case condition ii) is obviously satisfied, and i) holds since \( u_{icc} \) is negative and independent of \( \theta_i \).

In the following analysis, when we determine the sign of the comparative statics effects we shall focus our attention on equilibria which are standard to shocks.

### 3 Constrained inefficiency of competitive equilibria

#### 3.1 Feasible policies and allocations

As we discussed in the Introduction, it is important to specify the set of instruments available to a social planner to provide an economically meaningful definition of over or under investment. In this section, we shall consider instruments consisting in the direct control of capital and labor.

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Davila et al. (2005) consider the case where labor supply is inelastic and \( \rho_i^L (\theta_i) \) is constant, thus an equilibrium in their set up is automatically standard to shocks.
use of such instruments is clearly very demanding for a policy maker, and so they do not constitute policy tools of practical value. But the analysis of such case will provide some quite useful insights, and be instrumental for the subsequent analysis of the case where the policy tools are only given by linear, anonymous taxes on labor and capital.

More specifically, suppose the social planner can directly control the amounts of investment, \( k_i \), as well as of labor hours, \( h_i^0 \), for all consumers in every state \( \theta_i \). That is, the policy instruments available to the planner can be identified with a tuple \( \left( k_i, \left( h_i^0 \right)_{\theta_i \in \Theta_i} \right) \). All the other, non-policy variables are determined in competitive markets. Specifically, since the planner cannot control the firm’s decisions, the prices \( r \) and \( w \) are endogenously determined in equilibrium in such a way that the firm’s demand for inputs equal the aggregate supply of inputs set by the planner: \( r = F_K (K, L) \) and \( w = F_L (K, L) \) for 

\[
L = \frac{1}{2} \sum_{i=1}^{I} E \left[ \rho^L_\theta \left( \theta_i \right) h_i^0 \right], \quad K = \frac{1}{2} \sum_{i=1}^{I} \gamma_i k_i.
\]

We shall write \( r (K, L) \) and \( w (K, L) \) to indicate these maps associating the market clearing prices to the aggregate quantities, \( K \) and \( L \).

The levels of consumption of each type \( i \) consumer in the two periods and his leisure in the second period depend both on the levels of \( k_i \) and \( \left( h_i^0 \right)_{\theta_i \in \Theta_i} \), chosen by the planner and the associated market clearing prices. Specifically, first period consumption for type \( i \) is \( c_i^1 = e_i - k_i \) while in the second period in state \( \theta_i \) his leisure is \( \bar{h}_{i, \theta_i} \) and consumption is \( c_i^0 = w (K, L) L_i^0 + r (K, L) K_i^0 \). So the feasibility of a policy is naturally defined as follows:

**Definition 3** A policy \( \left( k_i, \left( h_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \) is said to be feasible if for every \( i \), we have \( e_i - k_i \geq 0 \), \( \bar{H}_i \geq h_i^0 \), and \( w (K, L) L_i^0 + r (K, L) K_i^0 \geq 0 \) at every \( \theta_i \), where \( r (K, L) \) and \( w (K, L) \) are the market clearing prices for \( K = \frac{1}{2} \sum_{i=1}^{I} \gamma_i k_i \) and \( L = \frac{1}{2} \sum_{i=1}^{I} E \left[ L_i^0 \right] \).

Clearly, if \( \left( \hat{w}, \hat{r}, \left( \hat{k}_i, \left( \hat{h}_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \right) \) is a competitive equilibrium, \( \left( \hat{k}_i, \left( \hat{h}_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \) is a feasible policy. By construction, the utility level of a type \( i \) consumer induced by a feasible policy \( \left( k_i, \left( h_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \) is given by:

\[
U_i \left( \left( k_i, \left( h_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \right) := v_i (e_i - k_i) + E \left[ u_i \left( w (K, L) L_i^0 + r (K, L) K_i^0, \bar{H}_i - h_i^0 \right) \right],
\]

where \( K = \frac{1}{2} \sum_{i=1}^{I} \gamma_i k_i \) and \( L = \frac{1}{2} \sum_{i=1}^{I} E \left[ L_i^0 \right] \). Following the common idea of second best analysis, we can present then a constrained efficiency notion

**Definition 4** A feasible policy \( \left( k_i, \left( h_i^0 : \theta_i \right) \right)_{i=1}^{I} \) is said to be constrained inefficient if there is a feasible policy \( \left( \tilde{k}_i, \left( \tilde{h}_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \) that is Pareto improving: \( U_i \left( \left( \tilde{k}_i, \left( \tilde{h}_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \right) \geq U_i \left( \left( k_i, \left( h_i^0 \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{I} \right) \) for every type \( i \), strictly for some \( i \), where \( U_i \) is defined as in (7).

We can similarly give a precise definition of over investment as characterizing situations where there exists a Pareto improving feasible policy such that \( \sum_i \tilde{k}_i < \sum_i k_i \) (and symmetrically for under investment).

---

\footnote{An equivalent notion could be stated for allocations instead of policies after defining a constrained feasible allocation as a consumption-leisure allocation achievable with a feasible policy.}
Note that, even though the planner can directly choose the level of individual investments and labor hours, the set of allocations attainable with feasible policies is still smaller than the set of feasible allocations considered in the standard Pareto efficiency notion. The planner is in fact still constrained by the requirement that the agents’ second period consumption respects the budget constraint \( r^i \rho^i = r \rho^K (\theta_i) k_i + w \rho^L (\theta_i) h^\theta_i \) for every \( i \) and \( \theta_i \), with prices \( r \) and \( w \) set at the competitive equilibrium level. Thus insurance against the consumers’ idiosyncratic shocks can only be provided by using the existing markets and the consumers’ investment and labor choices. Hence although a competitive equilibrium allocation is typically Pareto inefficient in the environment considered, it might still be constrained efficient in principle.

### 3.2 First Order effects and constrained inefficiency

From now on, fix a competitive equilibrium \( \left( \hat{w}, \hat{r}, \left( \hat{k}_i, \left( \hat{h}^\theta_i \right)_{\theta_i \in \Theta_i} \right)_{i=1}^I \right) \) which is standard to shocks.

The question we intend to ask is whether or not \( \left( \hat{k}_i, \left( \hat{h}^\theta_i \right)_{\theta_i \in \Theta_i} \right)_{i=1}^I \) is constrained efficient. If it is not, we want to see what kind of policies improve upon it and in particular whether or not there is over investment.

For this purpose, we shall study how the functions \( U_i \) behave around the equilibrium values, by differentiating them and evaluating them at the equilibrium values \( \left( \hat{k}_i, \left( \hat{h}^\theta_i \right)_{\theta_i \in \Theta_i} \right)_{i=1}^I \). Notice that a policy \( \left( k_i, \left( h^\theta_i \right)_{\theta_i \in \Theta_i} \right)_{i=1}^I \) has two effects on the expected utility level of a type \( i \) consumer given in (7): the first is of course the direct effect of the change in the values of \( k_i \) and \( \left( h^\theta_i \right)_{\theta_i \in \Theta_i} \); the second is an indirect effect due to the change in the values of the equilibrium prices \( r \) and \( w \). At a competitive equilibrium, however, the direct effect has no first order effect on welfare by the envelope property; in view of (3), the values \( \hat{k}_i \) and \( \left( \hat{h}^\theta_i \right)_{\theta_i \in \Theta_i} \) already maximize the utility of consumer \( i \) at the prices \( (\hat{w}, \hat{r}) \). Therefore, the only first order welfare effect of the policy change is the indirect effect, that is, only the pecuniary externality of the change in prices.

For this reason, as long as we are concerned with the derivative evaluated at an equilibrium allocation, we can take \( U_i \) in (7) as a function of \( K \) and \( L \) only, taking \( \left( \hat{k}_i, \left( \hat{h}^\theta_i \right)_{\theta_i \in \Theta_i} \right)_{i=1}^I \) as fixed constants. Let us calculate then its derivatives at the equilibrium values \( (\hat{K}, \hat{L}) \). Since the first period utility \( v_i \) does not depend on \( (K, L) \), we only need to differentiate the second period expected utility with respect to \( (K, L) \) taking \( \hat{L}^\theta_i \), \( \hat{K}^\theta_i \) and \( h^\theta_i \) as fixed at \( \hat{L}^\theta_i \) (= \( \rho^L (\theta_i) \hat{h}^\theta_i \)), \( \hat{K}^\theta_i \) (= \( \rho^K (\theta_i) \hat{k}_i \)) and \( \hat{h}^\theta_i \), respectively. Therefore, we have:

\[
\frac{\partial U_i}{\partial K} \bigg|_{(\hat{K}, \hat{L})} = \frac{\partial}{\partial K} \mathbb{E} \left[ u_i \left( w (K, L) \hat{L}^\theta_i + r (K, L) \hat{K}^\theta_i, \hat{H}_i - \hat{h}^\theta_i \right) \right] \bigg|_{(\hat{K}, \hat{L})} \\
= \mathbb{E} \left[ u_{ic} \cdot \frac{\partial}{\partial K} \left( w (K, L) \hat{L}^\theta_i + r (K, L) \hat{K}^\theta_i \right) \right] \bigg|_{(\hat{K}, \hat{L})} \\
= \mathbb{E} \left[ u_{ic} \cdot \left( \frac{\partial w}{\partial K} \hat{L}^\theta_i + \frac{\partial r}{\partial K} \hat{K}^\theta_i \right) \right],
\]

(8)

where \( u_{ic} \) is evaluated at the equilibrium levels of leisure and consumption in the second period, respectively \( \hat{H}_i - \hat{h}^\theta_i \) and \( \hat{w} \hat{L}^\theta_i + \hat{r} \hat{K}^\theta_i \), and \( \frac{\partial w}{\partial K} \) and \( \frac{\partial r}{\partial K} \) are both evaluated at \( (\hat{K}, \hat{L}) \). A similar
convention will be used throughout the paper. By definition, there is over investment in equilibrium if \(\frac{\partial u_i}{\partial K} < 0\) for every \(i\) at \((\hat{K}, \hat{\ell})\).

Similarly for labor, we have:

\[
\frac{\partial U_i}{\partial L}(K, \ell) = \frac{\partial}{\partial L} \mathbb{E} \left[ u_i \left( w(K, L) \hat{L}_i^{\theta_i} + r(K, L) \hat{K}_i^{\theta_i}, \hat{H}_i - \hat{h}_i^{\theta_i} \right) \right](K, L)
\]

\[
= \mathbb{E} \left[ u_{ic} \cdot \frac{\partial}{\partial L} \left( w(K, L) \hat{L}_i^{\theta_i} + r(K, L) \hat{K}_i^{\theta_i} \right) \right](K, L)
\]

\[
= \mathbb{E} \left[ u_{ic} \cdot \left( \frac{\partial w}{\partial L} \hat{L}_i^{\theta_i} + \frac{\partial r}{\partial L} \hat{K}_i^{\theta_i} \right) \right].
\]

Expressions (8) and (9) can be re-written in a more informative way as follows. Recall that \(F_K(K, L) = r(K, L)\) and \(F_L(K, L) = w(K, L)\). Hence \(\frac{\partial r}{\partial K} = F_{KK} < 0\) and \(\frac{\partial w}{\partial K} = F_{KL} > 0\). Moreover from the Euler equation, \(F_K(K, L) K + F_L(K, L) L = F(K, L)\), we obtain:

\[
\frac{\partial r}{\partial K} \cdot K + \frac{\partial w}{\partial K} \cdot L = 0. \tag{10}
\]

Similarly, we have \(\frac{\partial r}{\partial L} = F_{KL} > 0\) and \(\frac{\partial w}{\partial L} = F_{LL} < 0\), and

\[
\frac{\partial r}{\partial L} \cdot K + \frac{\partial w}{\partial L} \cdot L = 0. \tag{11}
\]

Coming back to the welfare change, taking (10) into account, we can decompose the marginal change in type \(i\)'s utility (8) as follows:

\[
\frac{\partial U_i}{\partial K}(K, \ell) = \mathbb{E} \left[ u_{ic} \cdot \left( \frac{\partial w}{\partial K} \hat{L}_i^{\theta_i} + \frac{\partial r}{\partial K} \hat{K}_i^{\theta_i} \right) - \left( \frac{\partial r}{\partial K} \cdot K + \frac{\partial w}{\partial K} \cdot L \right) \right] = \mathbb{E} \left[ u_{ic} \cdot \left( \hat{K}_i^{\theta_i} - \hat{\ell}_i \right) \right] + \mathbb{E} \left[ u_{ic} \cdot \left( \hat{\ell}_i - \hat{K}_i \right) \right] \frac{\partial r}{\partial K}, \tag{12}
\]

where all the variables are evaluated at the equilibrium.

In this decomposition, the terms \(\mathbb{E} \left[ u_{ic} \cdot \left( \hat{K}_i^{\theta_i} - \hat{\ell}_i \right) \right]\) and \(\mathbb{E} \left[ u_{ic} \cdot \left( \hat{\ell}_i - \hat{K}_i \right) \right]\) describe the relationship between the agent’s marginal utility and the idiosyncratic shocks. In what follows, we shall use a short-hand notation to refer to them:

\[
I_i^K := \mathbb{E} \left[ u_{ic} \cdot \left( \hat{K}_i^{\theta_i} - \hat{\ell}_i \right) \right], \tag{13}
\]

\[
I_i^L := \mathbb{E} \left[ u_{ic} \cdot \left( \hat{\ell}_i^{\theta_i} - \hat{K}_i \right) \right], \tag{14}
\]

where \(I\) stands for “insurance”. The reason is that these terms capture the component of the welfare effect of the change in prices that depends on how individual risks affect the agent’s consumption and leisure choices, that is on the extent by which such risks are insured. When such shocks are fully insured these terms are in fact zero.

**Lemma 3.** At an equilibrium which is standard to shocks, \(I_i^K \leq 0\) and \(I_i^L < 0\), i.e. the insurance effects defined in (13) and (14) are negative.

**Proof.** By the definition of the standard response to shocks, \(u_{ic}\) and \(\hat{K}_i^{\theta_i}, \hat{\ell}_i^{\theta_i}\) move in the opposite direction when \(\theta_i\) varies, hence \(\text{COV} \left[ u_{ic} \cdot \left( \hat{K}_i^{\theta_i} - \hat{\ell}_i \right) \right] \leq 0\) and \(\text{COV} \left[ u_{ic} \cdot \left( \hat{\ell}_i^{\theta_i} - \hat{K}_i \right) \right] < 0\).
Recall that for two random variables \( X \) and \( Y \), we have \( \mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) + \text{COV}(X,Y) \). Here \( \mathbf{E}\left(K_i^\theta - K_i\right) = 0 = \mathbf{E}\left(L_i^\theta - L_i\right) \) by construction, so the negative correlation implies that \( \mathbf{E}\left[u_{ic}\cdot\left(K_i^\theta - K_i\right)\right] \leq 0 \) and \( \mathbf{E}\left[u_{ic}\cdot\left(L_i^\theta - L_i\right)\right] < 0 \) hold. ■

The fact that \( I^K_i \leq 0 \) and \( I^L_i < 0 \) for every \( i \) tells us that the insurance effect associated to either a decrease in \( r \) or in \( w \) is a unanimous increase in individual welfare. To gain some economic intuition for this, notice that labor and capital constitute two alternative, ‘risky’ ways to provide wealth for future consumption. An increase in the market price of labor or of capital thus increases such risk and is so detrimental, the more so the riskier the instrument is.

A change in \( K \), however, has the opposite effect on factor prices \( w \) and \( r \). Hence the insurance effect of, say, an increase in \( K \) is given by the first and the third term in (12), which have respectively a positive and a negative sign, since \( \frac{\partial r}{\partial K} \leq 0 \) and \( \frac{\partial w}{\partial K} > 0 \). To determine which one prevails, notice that each of these terms will be smaller in absolute value the less random is the variable, \( \hat{K}_i^\theta \), or \( \hat{L}_i^\theta \), appearing in it, that is the less volatile is the return from the instrument considered to transfer wealth to the future. In particular, if \( \rho_{K}^i \) is a constant (i.e., there is no shock to capital accumulation as in the benchmark case) then \( I^K_i = 0 \), the insurance effect from the investment choice is zero. Consequently, since \( \frac{\partial w}{\partial K} > 0 \), a marginal increase in aggregate investment will reduce the induced utility of every household. So as far as the insurance effect is concerned, the households unanimously prefer a reduction of capital.

The remaining terms in (12), \( \mathbf{E}\left[u_{ic}\cdot\left(L_i - \hat{L}_i\right)\right] \), describe the relationship between the (expected) marginal utility of type \( i \) and the deviation of his average supply of capital and labor from the aggregate average supply. For future reference, we shall denote these terms as follows:

\[
D^K_i := \mathbf{E}\left[u_{ic}\cdot\left(K_i - \hat{K}\right)\right],
\]

\[
D^L_i := \mathbf{E}\left[u_{ic}\cdot\left(L_i - \hat{L}\right)\right],
\]

where \( D \) stands for “distribution”. They capture the effect of the price change on type \( i \)’s utility that is due to the relative size of his trades in the market with respect to those of the whole economy, that is to the ‘relative position’ of type \( i \) in the market. Evidently, when the economy consists of ex ante homogeneous types, these terms will be zero, hence their magnitude depends on the degree of heterogeneity among consumers in the economy at the equilibrium.

Summing up, we have the following decomposition result:

**Proposition 4** The first order effect on the welfare of type \( i \) consumers at a competitive equilibrium of a policy can be decomposed into an insurance effect and a distribution effect as follows:

\[
\frac{\partial U_i}{\partial K}(\hat{K},\hat{L}) = \{I^K_i + D^K_i\} F_{KK} + \{I^L_i + D^L_i\} F_{KL},
\]

\[
\frac{\partial U_i}{\partial L}(\hat{K},\hat{L}) = \{I^K_i + D^K_i\} F_{LK} + \{I^L_i + D^L_i\} F_{LL},
\]

where all terms are evaluated at the equilibrium values.

**Proof.** The expression of the derivative with respect to a change in \( K \) is obtained from (12) by substituting \( \frac{\partial r}{\partial K} \) and \( \frac{\partial w}{\partial K} \) with \( F_{KK} \) and \( F_{KL} \) and using (13) - (16). The next expression, for the change in \( L \), is analogously obtained, adding (11) to (9), collecting terms as in (12), using then (13) - (16) and replacing \( \frac{\partial r}{\partial L} \) and \( \frac{\partial w}{\partial L} \) with \( F_{LK} \) and \( F_{LL} \). ■
Remarq 1 The terms $I^K_i, D^K_i$, and $I^L_i, D^L_i$ describe the marginal effects on consumers’ utility of a unit change in prices, respectively $r, w$, at the equilibrium under consideration. The total marginal effect will be the sum of these terms multiplied by the marginal change in prices. In the present section, the marginal change in prices is induced by the direct control of $K$ and $L$, and hence is given by the second derivatives of the production function $F$ as we have seen in the decomposition result above. In the next sections various other policy tools are considered, leading to different forms for the marginal changes in prices. However, because of this common structure, the expression describing the welfare effects of such policies will be analogous to the one in Proposition 4: the same $I^K_i, I^L_i, D^K_i$, and $D^L_i$ appear with different terms describing the price change multiplying them.

Proposition 4 shows that the first order effect on agents’ welfare of a change in the aggregate supply of capital or labor consists of the weighted sum of the insurance and the distribution effects. The only difference between the effect of a change in capital and labor lies in the value of these weights, which are the derivatives of the production function. This point is better appreciated if we re-write the above decomposition expression using the Euler equations (10) and (11) as follows:

$$\frac{\partial U_i}{\partial K}(\hat{K}, \hat{L}) = \hat{K} F_{KK} \left\{ \left( \frac{I^K_i}{K} - \frac{I^L_i}{L} \right) + \left( \frac{D^K_i}{K} - \frac{D^L_i}{L} \right) \right\}, \quad (17)$$

$$\frac{\partial U_i}{\partial L}(\hat{K}, \hat{L}) = \hat{L} F_{LL} \left\{ \left( \frac{I^L_i}{L} - \frac{I^K_i}{K} \right) + \left( \frac{D^L_i}{L} - \frac{D^K_i}{K} \right) \right\}. \quad (18)$$

This decomposition result offers some clear insights on the relevant welfare effects. We give a few simple but interesting corollaries here. First of all, we observe that those consumers who favor a decrease in the stock of capital are exactly those who favor an increase in the amount of labor:

**Corollary 5** For any type $i$, $\frac{\partial U_i}{\partial K} \leq 0$ if and only if $\frac{\partial U_i}{\partial L} \geq 0$.

**Proof.** Compare (17) and (18): both $F_{KK}$ and $F_{LL}$ are negative by assumption, and the terms multiplying them are identical but with opposite sign.

Secondly, when the (absolute) magnitude of the distribution effects is bigger than that of the insurance effects, a reduction of the stock of capital benefits those types who invest more and work less than the economy average; vice versa an increase of capital.

**Corollary 6** When $|I^K_i| < |D^K_i|$, $6 \frac{\partial U_i}{\partial K} < 0$ holds if $\hat{K} - \hat{K} > 0$ and $\hat{L} - \hat{L} < 0$. When $|I^L_i| < |D^L_i|$, $\frac{\partial U_i}{\partial K} > 0$ if $\hat{K} - \hat{K} < 0$ and $\hat{L} - \hat{L} > 0$.

**Proof.** Note first that $\hat{K} - \hat{K} > 0$ and $\hat{L} - \hat{L} < 0$ imply that $D^K_i > 0$ and $D^L_i < 0$. Hence by Proposition 4 $\frac{\partial U_i}{\partial K} < 0$ follows when $|I^K_i| < |D^K_i|$, since $F_{KK} < 0$ and $F_{KL} > 0$. The second claim is established by a symmetric argument: when $\hat{K} - \hat{K} < 0$ and $\hat{L} - \hat{L} > 0$ we have $D^K_i < 0$ and $D^L_i > 0$ and hence $\frac{\partial U_i}{\partial K} > 0$ follows from $|I^L_i| < |D^L_i|$.

Finally, we show that, in the benchmark case, when consumers are all of the same type ($I = 1$), at an equilibrium that is standard to shocks we always have over investment and under supply of labor. In the absence of ex ante heterogeneity among consumers the distribution effects, whose signs are in general ambiguous, are zero. Although we do not establish it formally here, the result can be readily extended to economies where $I > 1$ but the ex ante heterogeneity across types is sufficiently small.

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6This condition is always satisfied when there is no shock to the productivity of the investment, in which case $I^K_i = 0$.

7This claim generalizes a result in Davila et al. (2005), who only consider the case of inelastically supplied labor.
Corollary 7 Assume that \( I = 1 \) and \( \rho^K_i(\theta_i) = \gamma_i \) for all \( \theta_i \in \Theta_i \). Then at an equilibrium which is standard to shocks, a reduction of the stock of capital (if \( \hat{K} > 0 \)) and/or an increase in the amount of labor (if \( \hat{L} < H_i \)) improve consumers’ utility. Thus an equilibrium which is standard to shocks is constrained inefficient and exhibits over investment.

Proof. When \( I = 1 \), \( \hat{K}_i = \hat{K} \) and \( \hat{L}_i = \hat{L} \) by construction, and so \( D^K_i = D^L_i = 0 \). Also \( \rho^K_i(\theta_i) = \gamma_i \) for all \( \theta_i \) implies that the insurance effect for capital vanishes as well: i.e., \( I^K_i = 0 \). So by Proposition 4, \( \frac{\partial U_i}{\partial K} = I^L_i F_{KL} \). This is negative since \( F_{KL} > 0 \) by the assumed properties of the technology and \( I^L_i < 0 \) by Lemma 3. Similarly for labor, assuming an interior solution, Proposition 4 shows that \( \frac{\partial U_i}{\partial L} = I^L_i F_{LL} > 0 \).

Remark 2 We have thus established that, in economies where consumers are ex ante identical, there is over investment in equilibrium when the idiosyncratic shocks only affect the productivity of labor. Even though this finding may appear in line with the one by Aiyagari (1995) the notion of over investment used here is quite different. Moreover the logic behind it is also different. To see this, notice that the result holds irrespectively of whether a precautionary motive is present or not, and in fact the level of the equilibrium interest rate plays no role in the above arguments. What is crucial is, on the other hand, the structure of the shocks: the result is completely overturned when the idiosyncratic shocks only affect the productivity of capital.

3.3 Social constrained optimality

The analysis in the previous section shows that, in the absence of ex ante heterogeneity among consumers, in the benchmark case (standard) competitive equilibria are constrained inefficient and exhibit over investment. When consumers are sufficiently heterogeneous on the other hand, we do not know whether a Pareto improvement can still be found only by modifying \( K \). This is because for some type \( i \) consumer the distribution effect may have the opposite sign and overturn the insurance effect, so that \( \frac{\partial U_i}{\partial K} > 0 \). If we consider a change both in \( K \) and \( L \), a welfare improvement exists if we can find weights \( \sigma^K \) and \( \sigma^L \), \( (\sigma^K, \sigma^L) \neq 0 \), such that the terms \( \sigma^K \frac{\partial U_i}{\partial K} + \sigma^L \frac{\partial U_i}{\partial L} \geq 0 \) have the same sign for every \( i \). Notice however that when \( I > 2 \) even this condition is not easily met. Indeed, we will see in our numerical example that the equilibrium may in fact be constrained efficient.

Remark 3 This is altogether in accord with the general constrained inefficiency result of Citanna - Kajii - Villanacci (1998): they show that with incomplete markets competitive equilibria can be Pareto improved (in terms of first order effects) if the planner has at least as many policy tools as the number of households plus one. In our framework, the number of policy tools which can have first order effects is effectively two, \( K \) and \( L \), whatever the number of households. So their analysis can be compared to ours only when \( I = 1 \), in which case we have indeed established constrained inefficiency (even with only one policy tool).

The above discussion reveals the difficulty of obtaining welfare improvements in the present framework when agents are ex ante heterogeneous. In what follows we shall consider the possibility of obtaining improvements when allocations are evaluated “under the veil of ignorance”, that is before the type of an agent is determined. The expected utility of an arbitrary consumer, who can be assigned to any type \( i \) with equal probability, is then given by \( W \left( \left( k_i, (h^0_i)_{\theta_i \in \Theta_i} \right)_{i=1}^I \right) := \sum_{i=1}^I \frac{1}{I} U_i \left( \left( k_i, (h^0_i)_{\theta_i \in \Theta_i} \right)_{i=1}^I \right) \). Some further justification for evaluating welfare in this way
comes from the view of the two period economy as a section of a dynamic economy where types are the result of past realizations of idiosyncratic shocks and individual decisions.

We shall refer to the problem of maximizing the social welfare function $W \left( \left( k_i, \left( h_i^\theta \right)_{\theta \in \Theta} \right)_{i=1}^I \right)$ by changing $K$ or $L$ as the constrained optimality problem, whereas constrained efficiency, as defined in the previous section, refers to the maximization of the vector of the utility functions of each individual type.

The derivative of the social welfare function $W$, evaluated at a competitive equilibrium, is simply the weighted sum of the derivatives of the individual utility functions found in Proposition 4 as well as (17) and (18):

$$\frac{\partial W}{\partial K} \bigg|_{(\bar{K}, \bar{L})} = \sum_i \frac{1}{I} \left[ \left\{ \{ I_i^K + D_i^K \} F_{KK} + \{ I_i^L + D_i^L \} F_{KL} \right\} \right],$$

$$= \bar{K} F_{KK} \sum_i \frac{1}{I} \left[ \left( \frac{I_i^K}{K} - \frac{I_i^L}{L} \right) + \left( \frac{D_i^K}{K} - \frac{D_i^L}{L} \right) \right].$$

$$\frac{\partial W}{\partial L} \bigg|_{(\bar{K}, \bar{L})} = \sum_i \frac{1}{I} \left[ \left\{ \{ I_i^K + D_i^K \} F_{LK} + \{ I_i^L + D_i^L \} F_{LL} \right\} \right],$$

$$= \bar{L} F_{LL} \sum_i \frac{1}{I} \left[ \left( \frac{I_i^L}{L} - \frac{I_i^K}{K} \right) + \left( \frac{D_i^L}{L} - \frac{D_i^K}{K} \right) \right].$$

**Definition 5** A competitive equilibrium exhibits **over investment** (from a social welfare perspective) if $\frac{\partial W}{\partial K} < 0$, and under investment if $\frac{\partial W}{\partial K} > 0$. Similarly, there is **under supply** of labor if $\frac{\partial W}{\partial L} > 0$ and over supply of labor if $\frac{\partial W}{\partial L} < 0$.

Although the sign of the distribution effect term may vary as noticed across types, the economy average $\sum_i \frac{1}{I} D_i^K$ and $\sum_i \frac{1}{I} D_i^L$ may still be signed. To see this, think of assigning a type $i$ to an agent at random; $\bar{K}_i$, $\bar{L}_i$, and $E[u_{ic}]$ can thus be regarded as random variables over states $i = 1, \ldots, I$ which are equally likely. Since $\sum_i (\bar{K}_i - \bar{K}) = 0$ and $\sum_i (\bar{L}_i - \bar{L}) = 0$, we have $\sum_i \frac{1}{I} D_i^K = Cov \left[ \bar{K}_i, E(u_{ic}) \right]$ and $\sum_i \frac{1}{I} D_i^L = Cov \left[ \bar{L}_i, E(u_{ic}) \right]$ by construction. We should expect that at a competitive equilibrium the relatively “rich” types of households, whose consumption level tends to be higher than the economy average, also tend to invest more than the average and work less than the average.\(^8\) This property relies on some normality of consumers’ demands and so we shall use again the term ’standard’ to refer to it:

**Definition 6** A competitive equilibrium is said be **standard in distribution** if $E[u_{ic}]$ is negatively correlated with $\bar{K}_i$ and positively correlated with $\bar{L}_i$.

When the equilibrium is standard both to shocks and in distribution, we simply call it a **standard equilibrium**. An immediate implication of the above property is as follows:

**Lemma 8** If $I > 1$, in an equilibrium standard in distribution the average distribution effect of a change in the price of capital $r$ is negative and that of a change in the price of labor $w$ is positive: i.e., $\sum_i \frac{1}{I} D_i^K < 0$ and $\sum_i \frac{1}{I} D_i^L > 0$.

Hence at an equilibrium that is standard in distribution, the average distribution effect of an increase in $K$ has a positive sign in (19). On the other hand, in the benchmark case the average\(^8\)This property holds for instance in the example we discuss later.
insurance effect has a negative sign in (19) since
\[ I^L_i F_{KL} < 0 \] for all \( i \) by Lemma 3. Thus there is a clear trade-off: whether there is under or over investment from a social welfare perspective depends on whether or not the average distribution effect prevails over the average insurance effect. Intuitively, the distribution effect gets magnified as the heterogeneity of income across types of households increases, whereas the insurance effect has no direct link to this heterogeneity. So we can expect that there is under investment in terms of ex ante welfare in an equilibrium with large income disparity, and indeed we will see this in numerical examples. We summarize this observation below:

**Proposition 9** Assume that \( \rho^K(\theta_i) = \gamma_i \) for all \( \theta_i \in \Theta_i \). A standard equilibrium exhibits under investment if the average distribution effect is larger than the average insurance effect in the sense that
\[
\sum_i |I^L_i| < \sum_i \left( \frac{D^K_i - D^L_i}{K} \right).
\]

**Proof.** The fact that the productivity of the investment is not subject to idiosyncratic shocks implies that \( I^K_i = 0 \) for every \( i \). Using (19), we obtain that \( \frac{\partial W}{\partial K} > 0 \) if and only if 
\[
\sum_i \left( -\frac{I^L_i}{T} + \left( \frac{D^K_i - D^L_i}{K} \right) \right) < 0.
\]
Since \( I^L_i < 0 \) by Lemma 3 and \( \sum_i D^K_i < 0 \) and \( \sum_i D^L_i > 0 \) by Lemma 8, under the assumed properties of the equilibrium, the claim follows. ■

**Remark 4** This result together with Corollary 7 show that, to determine whether or not there is over investment at an equilibrium, we should primarily look at the distribution of wealth across households. At an equilibrium where the income disparity is large enough, and hence the distribution effect is also large, we should expect that subsidizing capital is welfare improving. As already noticed in Remark 2, one can then deduce little concerning the direction of desirable policies from the observation that the level of the equilibrium interest rate is lower than with complete markets, that is if agents were able to trade in a complete set of contingent markets at the initial date.

**Remark 5** Notice that our local characterization results go through even when there are no shocks at all, so that markets are complete. In this special case, competitive equilibria are Pareto efficient and so a change in capital or labor will result in an efficiency loss. Ex ante welfare can be improved nonetheless, since redistribution from relatively rich to relatively poor types provides consumers with some insurance against ‘bad’ realizations of their type. So another interpretation of our results is that they clarify the nature of the trade-off between the efficiency properties of equilibrium allocations and the welfare properties arising from the inequality in income distribution.

## 4 Optimal Taxation

We analyzed so far whether agents’ welfare at a competitive equilibrium might be improved by suitably reducing/increasing the aggregate level of capital and labor when the available policy tools consist in the direct control of the levels of individual investment and labor supply. We turn now our attention to the case where such variables can only be indirectly controlled via anonymous, linear taxes on labor and capital. The net revenue of such taxes is then redistributed to consumers via lump sum taxes or transfers. In this case, consumers choose optimally the level of their investment and labor supply, while the firm still chooses the level of its inputs so as to maximize profits and prices \( r \) and \( w \) are set at a level such that markets clear.

Various scenarios can be considered concerning the specific definition of the taxes and subsidies. We shall study a few cases, which will clarify the essence of the optimal taxation problem in our context and its relationship with the decomposition found in Proposition 4.
Throughout this section, we shall fix a competitive equilibrium \( \left( \hat{w}, \hat{r}, \left( \hat{h}_i^\theta \right)_{\theta_i \in \Theta_i} \right) \), and study the first order effects of introducing a tax and subsidy scheme. In particular, we shall investigate whether capital and/or labor should be taxed from the point of view of social welfare.

### 4.1 Taxes on returns

We first look at the following tax/subsidy scheme. Denote by \( \tau_K \) the tax rate on the revenue from the investment in capital, and \( \tau_L \) the tax rate on labor income. When consumer \( i \) invests \( k_i \) and chooses \( h_i^\theta \) in each state \( \theta_i \), an amount \( w(1 - \tau_L)L_i^{\theta_i} + r(1 - \tau_K)K_i^{\theta_i} \) of his revenue next period in state \( \theta_i \) must be paid in taxes. The consumer also receives a lump sum transfer \( T_i(\theta_i) \) in state \( \theta_i \). The choice problem of a type \( i \) consumer is then modified as follows:

\[
\max_{k_i, \left( h_i^{\theta_i} \right)_{\theta_i \in \Theta_i}} v_i (e_i - k_i) + E \left[ u_i \left( w \left( 1 - \tau_L \right)L_i^{\theta_i} + r \left( 1 - \tau_K \right)K_i^{\theta_i} + T_i(\theta_i), \hat{H}_i - h_i^{\theta_i} \right) \right],
\]

(21)

where \( K_i^{\theta_i} \) and \( L_i^{\theta_i} \) are still as defined in (1) and (2). The maximization problem (21) remains a concave problem, and so the following first order conditions characterize its solutions:

\[
\begin{align*}
-v'_i (e_i - k_i) + E \left[ u_{ic} \cdot \rho^K_i(\theta_i) r \left( 1 - \tau_K \right) \right] &= 0, \quad (22) \\
u_{ic} \cdot w \left( 1 - \tau_L \right) \rho^L_i(\theta_i) - u_{il} &= 0, \quad \text{at every } \theta_i,
\end{align*}
\]

(23)

where the derivatives of \( u_i \) are evaluated at \( \left( w \left( 1 - \tau_L \right)L_i^{\theta_i} + r \left( 1 - \tau_K \right)K_i^{\theta_i} + T_i(\theta_i), \hat{H}_i - h_i^{\theta_i} \right)_{\theta_i \in \Theta_i} \).

In order to isolate the pure substitution effect of the tax, we shall consider first a tax-subsidy scheme which does not induce any redistribution of income across different types of agents nor even across realizations of the idiosyncratic state \( \theta_i \). That is, the tax subsidy scheme must satisfy the following budget balance condition for every realization of \( \theta_i \):

\[
w(1 - \tau_L)L_i^{\theta_i} + r(1 - \tau_K)K_i^{\theta_i} + T_i(\theta_i), \hat{H}_i - h_i^{\theta_i} = 0,
\]

(24)

for every \( i \). In other words, whatever an agent pays in taxes in any state \( \theta_i \), he also gets back as a lump sum transfer in that same state. The implementation of this tax scheme is clearly informationally rather demanding since it requires knowledge of individual trades and of the realization \( \theta_i \) of the idiosyncratic shocks, which may be private information of the agent. This scheme is thus not very realistic, but it allows to isolate the pure substitution effect of the tax and provides a useful theoretical benchmark for the subsequent analyses.

**Definition 7** An equilibrium with taxes and no redistribution nor insurance is a collection \( w, r, (\tau_K, \tau_L), \left( T_i(\cdot), k_i, (h_i^{\theta_i})_{\theta_i \in \Theta_i} \right)_{i=1}^I \) such that: i) for each \( i \), \( \left( k_i, (h_i^{\theta_i})_{\theta_i \in \Theta_i} \right) \) is a solution to (21), ii) profit maximization holds, i.e., \( F_K(K, L) = r, F_L(K, L) = w \), where \( K = \frac{1}{2} \sum_{i=1}^I \gamma_i k_i \) and \( L = \frac{1}{2} \sum_{i=1}^I E \left[ L_i^{\theta_i} \right] \), and iii) the budget balance (24) holds, for each \( \theta_i, i \).

An ex ante optimal tax scheme \( (\tau_K, \tau_L) \) is such that consumers’ ex ante welfare is maximized at the associated competitive equilibrium. Formally, such scheme solves the following problem: \(^{10}\)

\[
\max_{w, r, (\tau_K, \tau_L), \left( k_i, T_i, (h_i^{\theta_i}, K_i^{\theta_i}, L_i^{\theta_i}) \right)_{\theta_i \in \Theta_i}} \sum_{i=1}^I \left\{ v_i (e_i - k_i) + E \left[ u_i \left( w \left( 1 - \tau_L \right)L_i^{\theta_i} + r \left( 1 - \tau_K \right)K_i^{\theta_i} + T_i(\theta_i), \hat{H}_i - h_i^{\theta_i} \right) \right] \right\},
\]

\(^{10}\)The level of the lump sum transfers \( T_i(\cdot) \) is then uniquely determined by the budget balance condition (24).

\(^{10}\)The expression below should be multiplied by the constant \( 1/I \). Since this clearly plays no role in the analysis it is then omitted, both here and in what follows.
subject to the equilibrium conditions (22), (23), (24), conditions (1) and (2) defining $K_i^0$ and $L_i^0$, and the profit maximization conditions $w = F_L(K, L)$ and $r = F_K(K, L)$ evaluated at $(K, L) = (\sum_i \tau_i k_i / I, \sum_i E[L_i^0] / I)$. Equivalently, using (24) the objective function of this problem can be simplified as follows:

$$\sum_{i=1}^I \left\{ u_i (c_i - k_i) + E \left[ u_i \left( w L_i^0 + r K_i^0, H_i - h_i^0 \right) \right] \right\}.$$  \hspace{1cm} (25)

In what follows we shall denote this term as $W(\tau_K, \tau_L)$, to highlight the dependence of social welfare on the parameters describing the tax policy.

By construction, a competitive equilibrium is an equilibrium with taxes, where $\tau_K = \tau_L = 0$, and $T_i(\cdot) \equiv 0$ for every $i$. We intend to examine in particular whether $W(\tau_K, \tau_L)$ is increasing in $\tau_K$ and/or $\tau_L$ at $\tau_K = \tau_L = 0$. This will allow us to conclude that at least locally a positive tax on the realized return on capital/labor is welfare improving. To this end we assume the variables at an equilibrium with taxes are smooth functions of $(\tau_K, \tau_L)$ around $(\tau_K, \tau_L) = 0$.\textsuperscript{11} Hence, differentiating $W(\tau_K, \tau_L)$ and evaluating it at $\tau_K = \tau_L = 0$ we shall say that capital should be taxed if $\frac{\partial}{\partial \tau_K} W(\tau_K, \tau_L) > 0$ and it should be subsidized if $\frac{\partial}{\partial \tau_K} W(\tau_K, \tau_L) < 0$. A similar analysis can be done for labor.

Note that the envelope property from the individual optimization applies here again. Also note that the profit maximization condition must hold at any choice of tax rates, hence the Euler equation gives us relations analogous to (10) and (11):

$$\frac{\partial r}{\partial \tau_K} \cdot K + \frac{\partial w}{\partial \tau_K} \cdot L = 0, \hspace{1cm} (26)$$

$$\frac{\partial r}{\partial \tau_L} \cdot K + \frac{\partial w}{\partial \tau_L} \cdot L = 0. \hspace{1cm} (27)$$

We obtain a decomposition result for the effects of the introduction of the taxes which resembles our findings in Proposition 4 and equation (19), except that the changes in equilibrium prices are induced by a change in $\tau_K$ and $\tau_L$, not by a direct change of $K$ and $L$:

**Proposition 10** The (first order) welfare effects of the introduction of taxes with no redistribution nor insurance at a competitive equilibrium can be decomposed as follows:

$$\left. \frac{\partial W}{\partial \tau_K} \right|_{\tau=0} = \sum_i \left\{ (I_i^K + D_i^K) \frac{\partial r}{\partial \tau_K} + (I_i^L + D_i^L) \frac{\partial w}{\partial \tau_K} \right\}, \hspace{1cm} (28)$$

$$\left. \frac{\partial W}{\partial \tau_L} \right|_{\tau=0} = \sum_i \left\{ (I_i^K + D_i^K) \frac{\partial r}{\partial \tau_L} + (I_i^L + D_i^L) \frac{\partial w}{\partial \tau_L} \right\}, \hspace{1cm} (29)$$

where the terms $I_i^K$, $I_i^L$, $D_i^K$, and $D_i^L$ are still as in (13), (14), (15), and (16).

**Proof.** Since individual income in every state is not affected by the tax scheme, by the envelope property the only first order effect of the scheme on consumers’ utility is given by the change in equilibrium prices. Hence $I_i^\theta$, $K_i^\theta$, $h_i^\theta$, and $k_i$ can be treated as constants and $w$, $r$ as - differentiable

\textsuperscript{11}This will be generically the case at least if the underlying state space is finite. The number of equilibrium variables exceeds in fact the number of equations defining an equilibrium by two.
by assumption - functions of \((\tau_K, \tau_L)\). So we can do exactly the same operations as we did for the decomposition formula in Proposition 4 and (17) and (18), replacing \(\frac{\partial r}{\partial K}, \frac{\partial r}{\partial L}, \frac{\partial w}{\partial K}, \) and \(\frac{\partial w}{\partial L}\) with \(\frac{\partial r}{\partial \tau_K}, \frac{\partial r}{\partial \tau_L}, \frac{\partial w}{\partial \tau_K}, \) and \(\frac{\partial w}{\partial \tau_L}\), respectively.

This result shows that the welfare effects of taxation can be decomposed into the insurance and the distribution effects, whose properties have already been discussed in the previous section. But in order to determine the sign of the welfare effects in (28) and (29) we need to identify the signs of the changes in equilibrium prices. In general, prices could move in any direction, depending on the signs of the derivatives of the excess demand functions for capital and labor. We shall assume so that prices change in the natural direction: at the margin, an increase in the tax on the revenue from the sale of an input increases the gross revenue (i.e., cum tax) of the input but reduces the net revenue (i.e., net of tax). That is, we assume the following signs:

\[
\left. \frac{\partial r(\tau_K, \tau_L)}{\partial \tau_K} \right|_{\tau=0} > 0, \quad \left. \frac{\partial}{\partial \tau_K} \left[ (1 - \tau_K) r(\tau_K, \tau_L) \right] \right|_{\tau=0} < 0 \tag{30}
\]

\[
\left. \frac{\partial w(\tau_K, \tau_L)}{\partial \tau_L} \right|_{\tau=0} > 0, \quad \left. \frac{\partial}{\partial \tau_L} \left[ (1 - \tau_L) w(\tau_K, \tau_L) \right] \right|_{\tau=0} < 0 \tag{31}
\]

We shall refer to (30) and (31) as the natural signs for the changes in equilibrium factor prices.

We obtain a corollary similar to Corollary 5, which says whenever it is good to tax capital, it should be good to subsidize labor as well.

**Corollary 11** Assume the natural signs as above. Then taxing capital and taxing labor has opposite effects on welfare: \(\frac{\partial W}{\partial \tau_K} \geq 0\) if and only if \(\frac{\partial W}{\partial \tau_L} \leq 0\).

We can thus focus our attention on identifying the conditions under which capital should be taxed. The next result is an analogue of Proposition 9:

**Proposition 12** Assume the natural signs (30) and (31). Then capital should be taxed at a standard competitive equilibrium with taxes and no insurance nor redistribution if and only if there is over investment. Suppose, in addition, that \(\rho^K(\theta_i) = \gamma_i\) for all \(\theta_i \in \Theta_i\) and the competitive equilibrium is standard. Then capital should be taxed if \(\sum_i I_i > \sum_i \left( \frac{D^K_i}{K} - \frac{D^L_i}{L} \right) \) and subsidized if the reverse inequality holds; when \(I = 1\), capital should always be taxed.

**Proof.** Under (30), \(\frac{\partial r}{\partial \tau_K}\) has always the opposite sign of \(\frac{\partial r}{\partial K}\). Hence the same is true for the expression for \(\frac{\partial W}{\partial \tau_K}\) in (28) and that for \(\frac{\partial W}{\partial K}\) in (19), which establishes the first claim. Given this, the following claims are an immediate corollary of Proposition 9 and Corollary 7.

**Remark 6** Proposition 12 says that the idea of taxing an over used input is correct. And, as noticed in Remark 4, the determination of whether a positive tax is beneficial or not depends primarily on the comparison between insurance and distribution effects, not on the level of the equilibrium price of an input.

### 4.2 Lump-sum rebate as insurance

We consider next the case where there is still a linear tax on labor and capital income but the lump sum rebate is deterministic. The tax paid by a type \(i\) consumer equals \(w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i}\) in each state \(\theta_i \in \Theta_i\). By the i.i.d. assumption, the total, per capita tax paid by type \(i\) consumers is a deterministic amount, equal to \(w\tau_L L_i + r\tau_K K_i\), hence budget balance is still ensured with a deterministic lump sum rebate \(T_i\) satisfying:

\[
w\tau_L L_i + r\tau_K K_i = T_i, \tag{32}
\]
for every $i$. The rebate has in this case an insurance effect, as the difference between the tax paid and the rebate received is positive whenever the return on capital and labor exceeds its mean and negative otherwise. Although consumers’ types need to be observable for this scheme, the informational requirement is less demanding than in the previous case since the rebate is determined independently of the realization $\theta_i$ of the individual shock.

The choice problem of a consumer of type $i$ for this case is given as follows:

$$\max_{k_i \in [k^L_i, k^H_i]} v_i (e_i - k_i) + E \left[ u_i \left( w (1 - \tau_L) L_i^\theta + r (1 - \tau_K) K_i^\theta + T_i, \bar{H}_i - h_i^\theta \right) \right],$$

and an equilibrium with taxes and insurance but no redistribution can be defined, analogously to Definition 7, by suitably replacing the expression of the consumers’ objective function in (21) with the one above and the budget balance condition (24) with (32). By proceeding in the same way as in the previous section, we find that the expression of ex ante welfare constituting the objective function of the optimal taxation problem has now the following form:

$$\sum_i \left\{ v_i (e_i - k_i) + E \left[ u_i \left( w L_i^\theta + r K_i^\theta - \left\{ w \tau_L \left( L_i^\theta - L_i \right) + r \tau_K \left( K_i^\theta - K_i \right) \right\}, \bar{H}_i - h_i^\theta \right) \right] \right\},$$

(33)

and shall similarly denote it as $W^I (\tau_K, \tau_L)$, where the superscript $I$ highlights the new, insurance component.

Assume again the competitive equilibrium we are considering is a regular equilibrium in $(\tau_K, \tau_L)$, so that the equilibrium variables are smooth functions of $(\tau_K, \tau_L)$ around $(\tau_K, \tau_L) = (0, 0)$. We shall use again the superscript $I$ - e.g. $r^I (\tau_K, \tau_L), w^I (\tau_K, \tau_L)$ - to indicate that these functions are different from before as the equilibrium system is different. Differentiating $W^I (\tau_K, \tau_L)$ with respect to $\tau_K$ and $\tau_L$ and evaluating it at $\tau_K = \tau_L = 0$ yields the following expression:

$$\frac{\partial W^I}{\partial \tau_K} \bigg|_{\tau = 0} = \sum_i \left\{ \left( I_i^K + D_i^K \right) \frac{\partial r^I}{\partial \tau_K} + \left( I_i^L + D_i^L \right) \frac{\partial w^I}{\partial \tau_K} - E \left[ u_{ic} \cdot \left( \hat{K}_i^\theta - \hat{K}_i \right) \right] \right\}$$

$$= \sum_i \left\{ I_i^K \left( \frac{\partial r^I}{\partial \tau_K} - \hat{r} \right) + D_i^K \frac{\partial r^I}{\partial \tau_K} + \left( I_i^L + D_i^L \right) \frac{\partial w^I}{\partial \tau_K} \right\}$$

$$\frac{\partial W^I}{\partial \tau_L} \bigg|_{\tau = 0} = \sum_i \left\{ \left( I_i^K + D_i^K \right) \frac{\partial r^I}{\partial \tau_L} + \left( I_i^L + D_i^L \right) \left( \frac{\partial w^I}{\partial \tau_L} - \hat{w} \right) + D_i^L \frac{\partial w^I}{\partial \tau_L} \right\}.$$  

(34)

Comparing these expressions with the ones obtained in the previous section, (28) and (29), we see that there is an additional term in each of them. This is due to the fact that the expression of second period consumption in (33) differs from the one in (25) for the presence of the term $- \left\{ w \tau_L \left( L_i^\theta - L_i \right) + r \tau_K \left( K_i^\theta - K_i \right) \right\}$. The derivative of this additional term with respect to prices, when evaluated at $\tau_L = \tau_K = 0$, is zero - hence this term does not contribute to the price effect - but the derivative with respect to taxes is nonzero, $-w \left( L_i^\theta - L_i \right)$ and $-r \left( K_i^\theta - K_i \right)$ respectively. Hence the direct effects on agents’ utility of a change in taxes, at the margin, do not vanish, and they are equal to $-E \left[ u_{ic} \cdot \left( \hat{L}_i^\theta - \hat{L}_i \right) \right] \hat{w}$ and $-E \left[ u_{ic} \cdot \left( \hat{K}_i^\theta - \hat{K}_i \right) \right] \hat{r}$, which are nothing but the insurance effects, $I_i^L$ and $I_i^K$, multiplied by the opposite of the respective factor prices.

Since $I_i^K$ and $I_i^L$ are both negative by Lemma 3, we conclude that the additional term in both (34) and (35) is positive. Hence the claim in Corollary 11 is not valid in the present situation and it is possible that both the optimal tax on capital and that on labor are positive.

Somewhat in contrast to Proposition 12 we find that, under the same conditions, in the benchmark case the optimal tax on labor is always positive when tax rebates have an insurance role. On the other hand, the properties of the sign of the optimal tax on capital are unchanged.

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Proposition 13 Assume the natural signs\(^2\) (30), (31), and that \(\rho^i_L(\theta_i) = \gamma_i\) for all \(\theta_i \in \Theta_i\).
At a standard competitive equilibrium with taxes and insurance but no redistribution, labor should always be taxed while capital should be taxed whenever there is overinvestment, that is if \(\sum_i \left| \frac{\partial L_i}{\partial K_i} \right| > \left| \frac{\partial L_i}{\partial T} \right| \), and subsidized otherwise. If \(I = 1\), capital should always be taxed.

Proof. No productivity shock for capital implies \(I^K_i = 0\) for every \(i\). At a standard equilibrium, by Lemma 8 the average distribution effects are respectively negative and positive, \(\sum_i D^K_i < 0\) and \(\sum_i D^L_i > 0\), while by Lemma 3 the insurance effect \(I^L_i\) is negative for all \(i\). The natural sign assumption means then \(\bar{w} > \frac{\partial w_i}{\partial \tau_i} > 0\) and so also \(\frac{\partial r_i}{\partial \tau_i} < 0\) by the profit maximization conditions \(r = F_K, w = F_L\). Hence all the three terms in (35) are positive, which establishes the first claim.

Comparing (34) with (28), we see they differ only for the term multiplying \(I^K_i\). Since by assumption \(I^K_i = 0\) for all \(i\), the derivative with respect to \(\tau_K\) has the same form at an equilibrium without and with insurance. So the result follows from Proposition 12. \(\blacksquare\)

Remark 7 One might wonder why labor should be taxed even when there is (ex ante) under supply of labor. The argument of the proof of Proposition 9 implies that, when \(\sum_i \left| \frac{\partial L_i}{\partial K_i} \right| > \left| \frac{\partial L_i}{\partial T} \right| \), there is ex ante over investment and hence also, by Corollary 5, under supply of labor. But under (32) the lump sum tax rebate provides insurance against private idiosyncratic risks. In the benchmark case, where \(\rho^i_L(\theta_i) = \gamma_i\) for all \(\theta_i\), this insurance kicks in only if labor is taxed since the idiosyncratic shocks only affect labor productivity. The result in Proposition 13 shows that, under the natural sign assumption, the benefits from such direct insurance exceeds the welfare loss from further discouraging already under supplied labor.

4.3 Lump-sum rebate as insurance and redistribution

Next, we shall consider the case where the lump sum rebate is not only deterministic but also the same for all types. Then the per capita rebate equals the average tax payment across types:

\[
wt_L L + r\tau_K K = T. \tag{36}
\]

In this case the rebate has not only an insurance role, with respect to the individual shocks, but also a role of redistributing wealth across different types. Notice that this scheme requires no knowledge of the realizations of the individual shocks nor of individual types, and hence it is completely anonymous.

An equilibrium with taxes and insurance as well as redistribution is then similarly defined, by suitably replacing \(T_i(\theta_i)\) with \(T\) in (21) and the budget balance condition (24) with (36). The objective function of the optimal taxation problem becomes:

\[
\sum_i \left\{ v_i (e_i - k_i) + \mathbb{E} \left[ u_i \left( \epsilon^i, H_i - h^i_i \right) \right] \right\}, \tag{37}
\]

where for each \(i\), the second period consumption level is given by

\[
\epsilon^i_l = w L^i_l + r K^i_l - \left\{ w L_i \left( L^i_i - L_i \right) + \left( L_i - L \right) \right\} + r\tau_K \left( \left( K^i_i - K_i \right) + \left( K_i - K \right) \right), \tag{38}
\]

and will be denoted by \(W^{1R}(\tau_L, \tau_K)\) where \(R\) marks the new, redistribution element of the tax scheme.

Assuming again that the equilibrium variables are smooth functions of \((\tau_L, \tau_K)\) at \((0, 0)\),\(^{13}\) we

\(^{12}\)Strictly speaking, the natural sign assumption in the present framework should be stated by replacing \(\frac{\partial w}{\partial \tau}\) and \(\frac{\partial r}{\partial \tau}\) in (30) and (31) with \(\frac{\partial w}{\partial \tau}\) and \(\frac{\partial r}{\partial \tau}\) to reflect the fact that the equilibrium price maps are different. With a slight abuse of language we avoid to make this explicit, here and in what follows.

\(^{13}\)Equilibrium price maps are similarly denoted as \(w^{1R}(\tau_L, \tau_K), r^{1R}(\tau_L, \tau_K)\).
study under what conditions $W^{IR}(\tau_L, \tau_K)$ is increasing in $\tau_K$ and/or $\tau_L$ at $\tau_K = \tau_L = 0$, to conclude that a positive tax on the realized return on capital and/or on labor is welfare improving.

The expression of the derivatives of the social welfare function is now:

\[
\frac{\partial W^{IR}}{\partial \tau_K} \bigg|_{\tau=0} = \sum_i \left\{ (I_i^K + D_i^K) \frac{\partial u^i}{\partial \tau_K} + (I_i^L + D_i^L) \frac{\partial \hat{u}^i}{\partial \tau_K} - E \left[ u_{ic} \left( K_i^\theta - K_i \right) \right] \hat{\tau} - E [u_{ic}] (K_i - K) \hat{\tau} \right\} \\
= \sum_i \left\{ (I_i^K + D_i^K) \frac{\partial u^i}{\partial \tau_K} + (I_i^L + D_i^L) \frac{\partial \hat{u}^i}{\partial \tau_K} \right\} - \hat{\tau} \sum_i (I_i^K + D_i^K)
\]

(39)

\[
\frac{\partial W^{IR}}{\partial \tau_L} \bigg|_{\tau=0} = \sum_i \left\{ (I_i^L + D_i^L) \frac{\partial u^i}{\partial \tau_L} + (I_i^L + D_i^L) \left( \frac{\partial \hat{u}^i}{\partial \tau_L} - \hat{w} \right) \right\}
\]

(40)

They only differ from the corresponding terms in the previous section, (34) and (35), for the presence of an additional term in each of them, respectively $-\sum_i \{ E [u_{ic}] \left( K_i - K \right) \} \hat{\tau}$ and $-\sum_i \{ E [u_{ic}] \left( L_i - L \right) \} \hat{w}$. This is due to the fact that the expression for $c_i^\theta$ in (38) also has two additional terms, $-w\tau_L (L_i - L) - \tau K (K_i - K)$, which describe the redistributive component of the tax rebate. Differentiating them with respect to taxes and evaluating the effect on agents’ utility yields a positive average of these terms, constituting the new terms in (34) and (35), has respectively a positive and a negative sign. That is, the new redistributive effect of the tax scheme strengthens the case for taxing capital and weakens that for taxing labor.

**Proposition 14** Assume the natural signs (30), (31) and that $\rho^K(\theta_i) = \gamma_i$ for all $\theta_i \in \Theta_i$. At a standard competitive equilibrium with taxes and insurance as well as redistribution, both capital and labor should be taxed if $|\sum_i I_i^L| > |\sum_i D_i^L|$, i.e., the total distribution effect of labor is smaller than the total insurance effect of labor. Moreover, capital should be taxed if there is over investment.

**Proof.** Under the stated assumptions, for the same argument as in the proof of Proposition 13, we have again $I_i^K = 0$ and $I_i^L < 0$ for every $i$, and $\sum_i D_i^K > 0$ and $\sum_i D_i^L > 0$. Since the natural signs assumption implies $(\frac{\partial u^i}{\partial \tau_L} - \hat{\tau}) < 0$ and $(\partial \hat{u}^i/\partial \tau_K) < 0$, from (39) we see that $\frac{\partial W^{IR}}{\partial \tau_K} \bigg|_{\tau=0} > 0$ if $\sum_i (I_i^L + D_i^L) < 0$, i.e., $|\sum_i I_i^L| > |\sum_i D_i^L|$. Moreover, since the natural signs assumption also implies $(\frac{\partial u^i}{\partial \tau_L} - \hat{w}) < 0$ and $(\partial \hat{u}^i/\partial \tau_K) < 0$, from (40) we see that $\frac{\partial W^{IR}}{\partial \tau_L} \bigg|_{\tau=0} > 0$ follows from $\sum_i (I_i^L + D_i^L) < 0$. ■

Compare the condition, obtained from (39), for the positivity of the optimal tax on capital in the present framework, with the one for over investment we get from (19). Since both $F_{KK}$ and $\frac{\partial \tau_L}{\partial \tau_K} > 0$ we see the first condition is now weaker, thus the optimal tax on capital is always positive when there is over investment (e.g., when $I = 1$). The optimal tax on capital may also be positive when there is under investment (in contrast with Propositions 12 and 13).

Capital taxation, as we saw, is beneficial when the average insurance effect prevails over the average distribution effect. When the lump sum rebate is equal for all types, the tax has also a redistributive element, since wealthier consumers tend to have a higher income from capital. Hence the tax on capital effectively creates an income transfer from wealthier to poorer consumers, which is beneficial from the point of view of ex ante welfare.

We showed in Proposition 13 that taxing labor is beneficial when the tax rebate has an insurance effect. But when the rebate is equal for all types, and hence has an additional redistributive effect,
this effect works in the opposite direction since richer consumers tend to work less than poorer ones, and so the equal rebate effectively transfers income from the poor to the rich. It then follows that, if the distribution effect $\sum_i D_i$ is large enough, labor should rather be subsidized.

4.4 Taxing capital or labor?

In the previous analysis the effect of the tax on a factor’s income was combined with the effect of the lump sum rebate. The latter played an important role in the results, first because it allowed us to isolate the substitution effect of the tax and in the subsequent Sections 4.2, 4.3 because it generated additional effects, via the insurance and redistributional role of the rebate. The following natural question then arises: if the government cannot generate such lump sum transfers, should we tax capital or rather labor? The budget balance condition, in the absence of lump sum rebates, becomes:

$$w_\tau L + r\tau_K K = 0. \quad (41)$$

Therefore the two tax rates can no longer be independently set: if $\tau_K$ is positive $\tau_L$ has to be negative at the level needed to satisfy (41). The objective function of the optimal taxation problem in this case is directly obtained from the function $W^{IR}(\cdot)$ considered in the previous section, simply by setting $\tau_L = -\tau_K \frac{K^L}{L}$: therefore it is $W^{IR}(\tau_K, -\tau_K \frac{K^L}{L})$. Its derivative with respect to $\tau_K$ is then

$$\frac{\partial W^{IR}}{\partial \tau_K} = \frac{\hat{r}K}{wL} \sum_i \left( \hat{I}_i^L + D_i^L \right) = \frac{\hat{r}K}{wL} \sum_i \left( \left( \frac{I_i^L}{L} - \frac{K^L}{K} \right) + \left( \frac{D_i^L}{L} - \frac{D_i^K}{K} \right) \right),$$

we get

$$\frac{dW^{IR}}{d\tau_K} \bigg|_{\tau=0} = \hat{r} \left( \hat{r} \frac{\partial I^L}{\partial \tau_K} / \hat{r} + \frac{\partial w^{IR}}{\partial \tau_L} / \hat{w} - 1 \right) \sum_i \left( \left( \frac{I_i^L}{L} - \frac{K^L}{K} \right) + \left( \frac{D_i^L}{L} - \frac{D_i^K}{K} \right) \right). \quad (42)$$

Notice that the sum in the above expression is identical to the one appearing in (19) and has a negative sign if, and only if, we have ex ante under investment. The term premultiplying it, $\hat{r} \left( \hat{r} \frac{\partial I^L}{\partial \tau_K} / \hat{r} + \frac{\partial w^{IR}}{\partial \tau_L} / \hat{w} - 1 \right)$, turns out to be the effect of a marginal increase of the tax on capital on the net revenue $\hat{r}(1 - \tau_K)$ from the sale of capital. Indeed by (41) we have, at the margin,

$$\frac{\partial \tau_L}{\partial \tau_K} = -\left( \frac{\hat{r}K}{wL} \right)$$

and hence

$$\frac{d \left( r^{IR}(1 - \tau_K) \right)}{d\tau_K} \bigg|_{\tau=0} = \frac{\partial r^{IR}}{\partial \tau_K} - \frac{\partial r^{IR}}{\partial \tau_L} \left( \frac{\hat{r}K}{wL} \right) - \hat{r}, \quad (43)$$

where the second equality follows from the Euler equation and the profit maximization conditions, which imply: $\frac{\partial r}{\partial \tau_L} = -\frac{\partial w}{\partial \tau_L} \frac{L}{K}$.

One of the conditions stated in the natural sign assumption ((30) and (31)) says that, when the tax revenue for each individual is rebated with a lump sum transfer, the net revenue from the sale of a factor decreases if the tax on the factor increases. The condition that $\left( \frac{\partial r^{IR}}{\partial \tau_K} / \hat{r} + \frac{\partial w}{\partial \tau_L} / \hat{w} - 1 \right) < 0$ says the same property holds for the tax on capital in the present framework, where the revenue from such tax is rebated by a suitably defined subsidy on the sales of labor. The effect of such subsidy is shown by the second term appearing on the right hand side of (43), which has a positive

\footnote{In particular, from the last expressions in (39) and (40).}
sign under (31). The condition that \( \frac{\partial r}{\partial \tau} \hat{r} + \frac{\partial w}{\partial \tau} \hat{w} - 1 < 0 \) is then a little stronger than (30), and holds if factor prices are not too sensitive to the introduction of marginal taxes.

The previous discussion establishes the following result:

**Proposition 15** Assume that \( d\left[rIR(1 - \tau_K)\right]/d\tau_K \bigg|_{\tau=0} < 0 \). If no lump sum transfer is allowed, capital should be taxed if and only if there is under investment.

This result is somewhat surprising as the sign of the optimal tax on capital is exactly the opposite of what we found in Proposition 12 and, partly, also in Proposition 13. So when we are in the benchmark case, where there is no productivity shock for the investment in capital, and \( I = 1 \), capital should be subsidized even though we know there is over investment.

To gain some intuition for this result, recall first our previous finding that, when the only effect of taxes is the substitution effect of the price change, it is optimal to tax capital and to subsidize labor if and only if there is over investment. In the present environment, the tax scheme has no insurance or redistribution component, since there are no lump sum rebates, but the induced price change has also an income effect. The condition \( \left( \frac{\partial r}{\partial \tau} \hat{r} + \frac{\partial w}{\partial \tau} \hat{w} - 1 \right) < 0 \) says that the overall welfare effect is primarily determined by the income effect of the induced price change, which prevails over the substitution effect.

### 5 Numerical Example

In this section we consider a simple numerical example for which we derive the level of the optimal capital and labor tax rates for the different types of tax transfer schemes investigated in the previous section. In this framework we will also illustrate the findings of the local analysis carried out in the previous section, and compare them to the globally optimal tax rates. In principle, there is little reason to believe that the local information around the competitive equilibrium is sufficient to identify the properties of the optimal tax rates in the general set up we considered. But it will be seen that, under the functional forms and the specification of parameters which are commonly used in the literature, the results of the local analysis turn out to be useful to infer the properties of the global maximum.

There are two types of consumers and so \( I = 2 \). Consumers have the same preferences and second period endowments, as well as the same distribution of the idiosyncratic shock, they only differ for their initial endowments. We set \( e_1 > e_2 \) without loss of generality. So type 1 consumers are richer than type 2 consumers and we shall refer to the first ones as rich and the second ones as poor. There are two equally likely individual states, \( \Theta = \{ \theta_H, \theta_L \} \), and the common expected utility is

\[
U \left( c^0, (c^\theta, h^\theta)_{\theta \in \Theta} \right) = \frac{1}{1 - \sigma} \left( c^0 \right)^{1 - \sigma} + \mathbb{E} \left[ \frac{B}{1 - \sigma} \left( c^\theta \right)^{1 - \sigma} - \frac{\chi}{1 + \varphi} \left( h^\theta \right)^{1 + \varphi} \right]
\]

where \( c^0 \) is consumption in the first period, and \( c^\theta \) and \( h^\theta \) are consumption and labor supply in the second period in state \( \theta \). There is no shock to the productivity of investments in capital, as in the benchmark case, and the shock to the productivity of labor is the identity map:

\[
\rho^K_i(\theta) = 1, \quad \text{and} \quad \rho^L_i(\theta) = \theta \quad \text{for all} \quad \theta \in \Theta
\]

The production technology is Cobb-Douglas:

\[
F(K, L) = AK^\alpha L^{1 - \alpha}
\]
The values of the parameters are set as follows: $\sigma = 3$, $\varphi = 1$, $B = 8$, $\chi = 2.5$, $A = 1$, $\alpha = 0.36$. We also fix the average value of the labor productivity shocks, $\theta$, at unity, and the average initial endowment, $\bar{e} = (e_1 + e_2)/2$, at 3.7162,\textsuperscript{16} while allowing for different values for the magnitude of the shocks, identified by $\theta_H$, and the degree of heterogeneity, identified by $e_1/\bar{e}$. Note that under all parameter configurations considered, our example yields a standard equilibrium, and the sign conditions assumed in Propositions 12, 13, 14, and 15 are satisfied.

To start with, Figure 1 shows how market incompleteness affects capital accumulation and labor supply. There we fix the endowment distribution at $e_1/\bar{e} = 1.42$ and $e_2/\bar{e} = 0.58$, which implies a standard deviation of about 60 percent. Then we let $\theta_H$ vary from 1 to 1.4 with $\theta_L = 2 - \theta_H$. Thus the standard deviation of the idiosyncratic shock $\theta$ changes from zero to about 57 percent. The solid lines in the two panels of Figure 1 portray, respectively, the aggregate capital stock and the aggregate labor supply at the competitive equilibrium of the economy for the different values of the standard deviation of $\theta$. The dotted lines in the same panels depict also the aggregate capital and labor supply but at a competitive equilibrium where agents can trade in a complete market for contingent claims. We see that aggregate capital is greater with incomplete asset markets than with complete markets (due to the precautionary saving motive exhibited by the utility function considered). On the other hand, aggregate labor supply is lower, due to the income effect caused by the higher aggregate stock of capital. Moreover, the difference between the values with incomplete and complete markets is larger when the standard deviation of the idiosyncratic shock is also larger, that is, the uninsurable shock is more significant.

In the subsequent figures we fix the values of the idiosyncratic shocks at $\theta_H = 1.2$ and $\theta_L = 0.8$, and let the standard deviation of the distribution of the initial endowments, $e_i/\bar{e}$, vary from zero to 45 percent. Figure 2 plots the values of $\partial W/\partial \tau_K$ and $\partial W/\partial \tau_L$ evaluated at $\tau_K = \tau_L = 0$, that is, the marginal effects on ex ante welfare of introducing taxation on capital and labor income at the competitive equilibrium, as in the local analysis carried out in the previous section. Figure 3 plots the optimal tax rates, that is, the tax rates that maximize ex ante welfare $W(\tau_K, \tau_L)$. In both figures, we examine alternative specifications of the lump-sum transfers as considered in the previous section.

In each of the two figures the north-west panel corresponds to the tax scheme with no redistribution nor insurance discussed in Section 4.1 (see equation (24)). Under this tax scheme, as shown in Proposition 12, taxing capital or subsidizing labor is welfare enhancing (marginally at the competitive equilibrium), when the average distribution effect is relatively small. In the example here, the average distribution effect increases as the inequality in the initial endowments rises. This can be seen in the north-west panel of Figure 2: $\partial W/\partial \tau_K > 0$ and $\partial W/\partial \tau_L < 0$ when the inequality in initial distribution is sufficiently small, and vice versa when it is large. This local result is in accord with the values obtained for the optimal tax rates, displayed in the north-west panel of Figure 3. When the standard deviation of income distribution is close to zero the optimal tax rate on capital is around 2 percent. The optimal tax rate then decreases monotonically as inequality increases, becomes negative at the same point found in Figure 2 and equals approximately -3 percent when the standard deviation of income distribution is 45 percent. The reverse properties hold for the optimal tax on labor (whose level, in absolute value, is a bit lower).

The second tax scheme we consider is the one with insurance but no redistribution examined in Section 4.2 (see equation (32)). The north-east panel of Figure 2 plots again the marginal effects of

\textsuperscript{16}The values of $\sigma, \varphi, \alpha$ are in line with those commonly used in the macroeconomics literature. The value of $\bar{e}$ has been chosen so that, when the consumers are identical and there are no shocks, $e_1 = e_2$ and $\theta_H = \theta_L$, the equilibrium level of consumption is the same in the two periods, as in the steady state of a dynamic economy.
capital and labor taxes, illustrating the claim in Proposition 13. The sign of the marginal effect on ex ante welfare of capital taxation at the competitive equilibrium is the same as in the previous tax scheme.\textsuperscript{17} This is in accord with this proposition, since in our example there is no idiosyncratic shock to the return on capital. Also, the marginal effect of labor taxation is now positive regardless of the degree of inequality in the initial endowment distribution. The north-east panel of Figure 3 shows how the optimal tax rates vary with the inequality in the initial endowment. With no inequality the optimal tax rates on capital and labor are both positive, equal respectively to around 6 and 10 percent. As the degree of inequality increases, the optimal tax rate on capital decreases. Note however that it stays positive even when the marginal effect of capital taxation at an equilibrium with no taxes becomes negative. This is the region where the suggestions provided by the local analysis turns out to be misleading for the globally optimal tax rates, but the size of the region appears to be small. The optimal tax rate on labor is positive and increases with the degree of inequality, in accord with what suggested by the local analysis, reaching a level of around 12 percent when the standard deviation of the endowment distribution is 45 percent.

The third tax scheme we consider is the one with insurance as well as redistribution, discussed in Section 4.3 (see equation (36)). The south-west panel of Figure 2 plots the marginal effects of taxation for this case, which illustrate the claim in Proposition 14. The marginal welfare effect of capital taxation turns out to be always positive and, in contrast with the previous cases, to increase with the degree of initial inequality as we see in Figure 2. So in this case, in the economy considered in this example capital should be taxed even when there is under investment. We see in Figure 2 that the effect of labor taxation is also positive but falls as the degree of inequality increases, reflecting the fact that the distribution effect works against taxing labor. The south-west panel of Figure 3 shows that the optimal capital and labor tax rates are again in line with what the local analysis suggests: the optimal tax rate on capital income is positive and increases with the degree of inequality, reaching a level of around 35 percent when the standard deviation of income is the highest considered. The optimal tax rate on labor income is also positive though considerably lower and declines slightly with the degree of inequality. As argued in Section 4.3, the nature of the tax rebate in this case strengthens the case for taxing capital.

The last tax scheme we consider is the one without lump-sum transfers, analyzed in Section 4.4 (equation (41)). The south-east panel of Figure 2 plots the marginal effects of taxation for this case. By construction, the tax rates on capital and labor income move in opposite directions to balance the government’s budget. By comparing the south-east with the north-east panels we see that under this tax scheme the sign of the marginal effect of capital taxation is exactly the opposite to the one found for the tax schemes without insurance nor redistribution analyzed in Section 4.1, in accord with what shown in Proposition 15. Thus the marginal effect of capital taxation on ex ante welfare is negative with no inequality in the initial endowment, but increases as inequality rises and becomes positive for a sufficiently large degree of inequality. In the south-east panel of Figure 3 we confirm that the optimal tax rates on capital and labor behave as our local analysis suggests: the optimal tax on capital ranges from -10 to 10 percent as the standard deviation of income distribution varies from 0 to 45 percent.

6 Concluding remarks

Our results demonstrate that the optimal tax rate on capital may be negative. Moreover this finding applies for equilibria exhibiting standard properties and hence does not rely on any pathological

\textsuperscript{17}The values are different since the derivatives of equilibrium price functions are different.
properties, as the existence of upward sloping demand curves. In these cases, by subsidizing capital and taxing labor the level of capital accumulation increases beyond the already “excessively high” level of the equilibrium with no taxes.

As we have pointed out, the determination of the optimal taxation level is a second best problem, constrained by the fact that the attainable allocations are those obtained as competitive equilibria for a suitably designed tax system. Hence a simple comparison to the “efficient level” of capital in the complete market equilibrium does not provide a right intuition for whether or not capital is excessive.

In the environment considered competitive equilibria are generally Pareto inefficient. The usual textbook argument suggests that the inefficiency can be decomposed into two parts: the allocational inefficiency and the production inefficiency. The latter is caused by over/under use of capital/labor in our model. If the capital/labor ratio is above the efficient, complete market level a positive tax on capital should move this ratio towards its efficient level, thus generating an economic surplus. This would make the agents better off if the surplus is distributed among the agents appropriately.

This line of argument, while correct in a partial equilibrium analysis, is insufficient in a general equilibrium environment with incomplete markets for at least two reasons. First, it ignores the fact that the introduction of taxes may modify equilibrium prices and, with incomplete markets, the level of equilibrium prices affects the consumers’ ability to hedge the risk they face. Hence taxes, by modifying such prices, may improve this ability, i.e., the allocational efficiency may be improved at the expense of production efficiency. It is indeed the insurance effect in our analysis what captures this intuition.

Secondly, the surplus can only be distributed via taxes and lump sum rebates as well as price changes, so an appropriate distribution of the surplus may not be feasible. Taxes inevitably induce some income redistribution, indirectly through price changes and directly through the distribution of the tax revenue. Such distribution effects, as we saw in our analysis, will be different and have opposite signs for agents with sufficiently different income levels and preferences. That is, the taxation scheme we consider is not a perfect instrument to distribute the economic surplus from the increase in production efficiency among the different agents.

Furthermore, the social welfare criterion we consider (‘under the veil of ignorance’) favors equality in income. Hence social welfare may be improved by increasing income equality while sacrificing production efficiency, and this is true even when markets are complete.

The answer to the question of what is the optimal taxation level with incomplete markets is a delicate one, and one cannot deduce it from the partial equilibrium intuition. It is our contribution to identify the insurance and distribution effects of the tax and the determinants of the sign and magnitude of these effects. This in turn provides useful information about the sign and magnitude of the optimal tax on capital and labor, in a general equilibrium setting.

Finally, we readily acknowledge that while we are able to decompose and identify the effects of taxes in a clear manner, the consideration of a two period environment when examining taxes on capital is a limitation. In a companion paper (Gottardi, Kajii and Nakajima (2011)) we examine an infinite horizon environment: assuming that agents are identical ex ante, the income distribution at a particular time period is then the result of past realizations of the uncertainty. We find that the main intuitions developed in this simple two period setup go through and allow to better understand the properties of optimal taxes also in these dynamic economies.
References


Appendix

**Proofs for Lemmas 1 and 2.** Consider the second-period utility maximization problem for a type $i$ consumer at $\theta_i$:

$$\max_{c,l} u_i(c, l)$$

subject to

$$c + \rho^{L}_i(\theta_i)l = \rho^{K}_i(\theta_i)k_i + \rho^{r}_i(\theta_i)\bar{H}_i,$$
where \( k_i \) has already been chosen in the first period. This problem can be restated as a standard consumer problem in general equilibrium:

\[
\begin{align*}
\max_{c,l} & \ u(c,l) \\
\text{subject to} & \ c + pl = m + pH,
\end{align*}
\]

where \( p = w\rho^L(\theta_i) \) and \( m = r\rho^K(\theta_i)k_i \) are taken as given. Writing \( c(p,m) \) and \( l(p,m) \) for the derived demand functions for the consumption good and leisure respectively, and denoting by \( \lambda(p,m) \) the Lagrange multiplier, the following first-order conditions characterize the demand functions:

\[
\begin{align*}
\frac{u_c(c(p,m),l(p,m)) - \lambda(p,m)}{\partial p} &= 0, \\
\frac{u_l(c(p,m),l(p,m)) - \lambda(p,m)}{\partial l} &= 0, \\
\frac{-(c(p,m) + pl(p,m))}{\partial m} &= -(m + pH).
\end{align*}
\]

Recall that \( \rho^L \) and \( \rho^K \) are comonotonic, and so are both \( p \) and \( m \). Therefore, to establish Lemma 1, it suffices to show that \( \lambda(p,m) \) is decreasing in \( m \) and \( p \). Since \( \lambda(p,m) \) is the derivative of the indirect utility function with respect to income and the indirect utility function is concave in income for a concave utility, it readily follows that \( \lambda \) is decreasing in \( m \).

We shall now show that \( \lambda(p,m) \) is decreasing in \( p \) as well. To simplify the notation, we shall omit reference to \( (p,m) \) below. To find the derivatives of \( \lambda \) (as well as those for \( c \) and \( l \)), we follow the standard technique of differentiating the system of the first order conditions:

\[
\begin{bmatrix}
\frac{\partial u_c}{\partial p} & u_{cl} & -1 \\
\frac{\partial u_l}{\partial p} & u_{ll} & -p \\
-1 & -p & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\lambda \\
l - H
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_c}{\partial p} \\
\frac{\partial u_l}{\partial p} \\
\frac{\partial \lambda}{\partial p}
\end{bmatrix}.
\]

The strict concavity assumption implies that the determinant of the square matrix above is positive:

\[
\Delta := -u_{cc}p^2 + 2u_{cl}p - u_{ll} > 0,
\]

and we have:

\[
\begin{bmatrix}
\frac{\partial u_c}{\partial p} & u_{cl} & -1 \\
\frac{\partial u_l}{\partial p} & u_{ll} & -p \\
-1 & -p & 0
\end{bmatrix}^{-1}
= \frac{1}{\Delta}
\begin{bmatrix}
-p^2 & p & u_{ll} - pu_{cl} \\
p & -1 & u_{cc}p - u_{cl} \\
-u_{ll} - pu_{cl} & u_{cc}p - u_{cl} & u_{cc}u_{ll} - (u_{cl})^2
\end{bmatrix}.
\]

Thus we have:

\[
\frac{\partial \lambda}{\partial p} = \frac{1}{\Delta}
\left((u_{cc}p - u_{cl})\lambda + (u_{cc}u_{ll} - (u_{cl})^2)(l - H)\right),
\]

which is negative. Indeed, using the first order condition, \( (u_{cc}p - u_{cl})\lambda = u_{cc}u_{ll} - u_{cl}u_{c} < 0 \) where the inequality holds by the normality of consumption good, and \( (u_{cc}u_{ll} - (u_{cl})^2) > 0 \) by concavity and \( (l - H) < 0 \). Therefore, Lemma 1 has been established.

Notice that the labor supply in efficiency units corresponds to \( (H - l)p/w \) in the consumer problem above, so in order to establish Lemma 2 it suffices to show that \( (H - l)p \) is increasing in \( p \). From the system of equations above,

\[
\frac{\partial l}{\partial p} = \frac{1}{\Delta}
\left(-\lambda + (u_{cc}p - u_{cl})(l - H)\right),
\]

28
and so
\[
\frac{d}{dp} \left( (\bar{H} - l) p \right) = (\bar{H} - l) - p \frac{\partial}{\partial p} l,
\]
\[
= (\bar{H} - l) - \frac{p}{\Delta} (-\lambda + (u_{cc}p - u_{cl})(l - \bar{H})) ,
\]
\[
= (\bar{H} - l) \left(1 + \frac{p}{\Delta} (u_{cc}p - u_{cl})\right) + \frac{p\lambda}{\Delta}.
\]

Now,
\[
1 + \frac{1}{\Delta} \left( u_{cc}p^2 - pu_{cl} \right) = \frac{1}{\Delta} \left( (-u_{cc}p^2 + 2u_{cl}p - u_{ll}) + (u_{cc}p^2 - pu_{cl}) \right),
\]
\[
= \frac{1}{\lambda\Delta} (u_{cl}u_{l} - u_{c}u_{ll}),
\]
\[
> 0,
\]
where the last inequality follows from the normality of leisure. This proves Lemma 2.
Figure 1: Capital stock and labor supply with complete and incomplete markets. The horizontal axis measures the standard deviation of the labor productivity shock (percent).
Figure 2: Marginal effects on ex ante welfare of taxation on capital and labor under alternative specifications of the lump sum rebates. The horizontal axis measures the standard deviation of the initial endowment distribution (percent).
Figure 3: Optimal tax rates on capital and labor under alternative specifications of the lump sum rebates. The horizontal axis measures the standard deviation of the initial endowment distribution (percent).