# Decentralizing Efficient Allocations in Economies with Adverse Selection: The General Case

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## Abstract

We study competitive economies with adverse selection and fully exclusive contractual relationships. We show that Walrasian equilibria always exist and are efficient for the general class of adverse selection insurance economies considered by Prescott and Townsend (1984). The result requires an appropriate expansion of the set of markets, in the spirit of Arrow (1969) and Lindahl (1919), to include markets for consumption rights so as to internalize the externality induced by the incentive constraints with adverse selection. Given the non-convexities generated by these constraints, the commodity space is enlarged to allow for lotteries. Our analysis has then also some useful implications for the study of Arrow-Lindahl equilibria in general economies in the presence of non-convexities.

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# 1. Introduction and Motivation

We study competitive economies with adverse selection and fully exclusive contractual relationships. In particular, we consider pure exchange economies where agents trade to insure against the uncertainty affecting their endowments. Uncertainty is purely idiosyncratic. Agents have private information regarding the probability distribution of their endowments as well as about their preferences. Insurance firms, who sell contracts to consumers, can enforce exclusive contractual relationships through the full monitoring of individual trades.

Various equilibrium concepts have been introduced to study such economies. The standard strategic analysis, due to Rothschild and Stiglitz (1976) and Wilson (1977), considers the Nash equilibria of a game in which insurance firms choose which exclusive contract to issue. The competitive aspect of the market is captured, in strategic models, by the free entry of insurance firms. We study instead Walrasian equilibrium concepts, where agents and insurance companies act as price takers in competitive markets for contracts and all admissible contracts are available for trade, at the market price.

Walrasian equilibria of economies with asymmetric information and exclusive contractual relationships have been earlier considered in a seminal paper by Prescott and Townsend (1984). Primary objective of that paper is to extend Arrow and Debreu competitive analysis to economies with asymmetric information. For moral hazard economies Prescott and Townsend show that, in the presence of a complete set of contingent markets and with suitable restrictions on admissible trades of individuals, Walrasian equilibria exist and are incentive efficient. However, their approach does not provide any conclusive result in the case of adverse selection economies.

In this paper we show that existence and efficiency of competitive equilibria can be established for economies with adverse selection as well. Our main focus, as Prescott and Townsend's, is on the analysis of market structures and institutional environments that allow to decentralize incentive efficient allocations. Thus the only limitations on the markets which can be open and the restrictions which can be imposed on individual trades are those due to the unobservability of agents' types and characteristics.[1]

Since the odds of individual states are type specific, completeness of markets requires that markets, and prices, are differentiated according to the agents' type. But then the private information of agents' types implies that only incentive compatible trades can be admissible, and the level of trades chosen by agents of one type influences, via its effect on the incentive compatibility constraints, the admissible trades of agents of other types. A specific form of consumption externality arises so in economies with adverse selection and exclusive contracts. We show that such externality can be internalized by introducing a properly enlarged structure of markets, in the spirit of the approach

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[1]In this respect we depart from other, recent work on Walrasian equilibria of economies under adverse selection, as Dubey and Geanakoplos (2002), Gale (1992, 1996), Bisin and Gottardi (1999).

pioneered by Arrow (1969) and Lindahl (1919), that includes markets for consumption rights.

More precisely, we show that the incentive efficiency of competitive equilibria can be obtained with the following structure of markets. There is a separate market designated for each type. Any agent chooses then first in which market to trade, effectively announcing his type. In this market the agent buys consumption claims, contingent on every possible realization of the individual uncertainty, as well as consumption rights for each of the other types which can be announced by agents. An enforcement mechanism is in place, requiring that agents cannot purchase consumption claims unless they also buy an *appropriate* amount of consumption rights. What is *appropriate* is in turn determined by the incentive compatibility constraints: the bundle of consumption claims acquired in the market designated for type $i$ agents must be incentive compatible with respect to the amount of consumption rights purchased for every other type $j \neq i$. The trades in the markets for consumption rights allow to generate cross-subsidies among types, which may be needed to sustain incentive efficient allocations. Furthermore, the specification of the set of admissible trades in each market must be such that no other type than the designated one wishes to trade in that market.

This paper is a complement to Bisin and Gottardi (2006), where the existence and incentive efficiency of Walrasian equilibria is established for a special class of adverse selection insurance economies, the one considered by Rothschild and Stiglitz (1976). Rothschild and Stiglitz economies are characterized by the fact that there are two types of agents, and the type only affects the probability distribution of the agents' endowment, all agents have the same preferences. These economies are special primarily because non-convexities do not matter: agents' demands are in fact deterministic, continuous functions and no randomization occurs at any efficient allocation.

In this paper we study instead the general class of adverse selection economy considered by Prescott and Townsend (1984), where there is an arbitrary finite number of types and the type affects both preferences and endowments. Agents may thus exhibit different income patterns as well as different risk attitudes. We show that Walrasian equilibria always exist and are efficient. Showing the existence of Walrasian equilibria in this general framework poses some new problems. The restriction imposed by the incentive compatibility constraints in the specification of the set of admissible trades of every agent implies in fact that such set is non-convex, and non-convexities cannot be ignored in the general case. We are therefore naturally lead to expand the commodity space to include lotteries as in Prescott and Townsend (see also Kehoe, Levine and Prescott, 2002). Standard arguments for establishing existence of equilibria with lotteries need however to be amended for our economy. Because of the peculiar nature of the structure of markets considered, and in particular the presence of markets for consumption rights, it is not sufficient in our economy that the market clearing conditions hold *on average*, as usually the case in large economies when we allow for lotteries. In particular, a consistency condition must hold in markets for consumption rights, whereby the bundle of

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rights chosen has to be exactly equal (that is, for each point of the support of the lottery, and not only on average) for all agents' types. A related difficulty arises to show the existence of Arrow-Lindahl equilibria for general economies with externalities or public goods in the presence of non-convexities (as far as we know, still an open problem).

Recently, Rustichini and Siconolfi (2008) have also studied competitive economies with adverse selection. They found robust instances of non-existence of Walrasian equilibria. While the notion of equilibrium used in their paper differs from ours in several details, the main source of the difference in the results can be identified in the fact that Rustichini and Siconolfi impose weaker restrictions on the set of admissible trades in each market, which do not ensure that agents always self select into the market designated for their own type.[2] While the restrictions we impose on admissible trades in the various markets do not need any information on agents' types to be implemented, they nonetheless require markets to internalize a strong form of incentive compatibility. It is an implication of the analysis of Rustichini and Siconolfi (2008) that the structure of markets considered here, with markets for consumption rights to ensure efficiency, is not viable without these restrictions.

While we do not expose this formally in the paper, it is apparent that our findings have also important implications for the properties of Arrow-Lindahl equilibria in general economies. When agents' types are non observable, as shown by Roberts (1979), Arrow-Lindahl equilibrium allocations may fail to be incentive compatible. In such case however, we could ensure incentive compatibility by imposing appropriate restrictions on agents' admissible trades, as done here. The existence of incentive efficient equilibria could then be showed along similar lines as the ones followed in this paper.

The paper is organized as follows. The next section describes the economy and define incentive efficient allocations. In Section 3 the structure of markets is introduced and the main results, the existence and efficiency of Walrasian equilibria, are established.


## 2. The Economy

There is a continuum of agents of $I$ different types. Types differ both according to their endowments and preferences and an agent's type is only privately observed by him. The fraction of agents of type $i \in I = \{1, .., I\}$ in the population is denoted by $\xi^i$; $\xi^i > 0$ and $\sum_{i \in I} \xi^i = 1$.

We restrict our attention to economies with a single consumption good.[3] Uncertainty enters the economy via the level of the individual endowment. There are $S$ possible states for every individual. His endowment when $s \in S$ is realized is given by $\omega(s)$;

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[2]See also Hammond (2005) for another attempt at designing a market mechanism allowing to decentralize constrained efficient allocations in adverse selection economies.

[3]This is primarily to simplify the notation. It should be clear from the argument below that the extension to a finite number of commodities is straightforward.

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$$\omega \equiv \begin{bmatrix} \vdots \\ \omega(s) \\ \vdots \end{bmatrix} \in \mathbb{R}^S_+.$$ Let $\pi^i(s)$ denote the probability that individual state $s \in S$ is realized for an agent of type $i \in I$. The type of every agent affects so the probability distribution, not the support ($\omega$), of his endowment. The random variables describing the individual endowments are independently distributed across all agents (and identically distributed across agents of the same type). Uncertainty is thus purely idiosyncratic.

The preferences of each agent of type $i \in I$ are described by a von Neumann-Morgernstern utility function, with utility index $u^i : \mathbb{R}_+ \to \mathbb{R}$, defined over consumption in each (individual) state $s \in S$. Preferences then also depend on the agent's type.

We assume, with no loss of generality, that $\omega(s) < \omega(s')$ if $s < s'$. Moreover:

**Assumption 1.** *Endowments are strictly positive, in all states (and for all agents):* $\omega(s) > 0$, *for all* $s \in S$. *For all* $i \in I$, $u^i(.)$ *is strictly monotonic, strictly concave, twice continuously differentiable, and such that* $\lim_{v \to 0} (u^i)'(v) = \infty$.

The economy described is essentially the same as the adverse selection insurance economy considered by Prescott and Townsend (1984).

The consumption level of an agent of type $i$, $i \in I$, in each of the agent's individual states is described by the vector $x^i \equiv (x^i(s))_{s \in S} \in \mathbb{R}^S_+$. To deal with the non-convexities generated by the incentive compatibility constraints, the set of possible consumption levels is expanded to allow for lotteries, defined over the following compact subset of $\mathbb{R}^S_+ : \mathbf{X} \equiv \left\{ x \in \mathbb{R}^S : 0 \le x(s) \le M, \ s \in S \right\}$, where $M$ should be viewed as arbitrarily large.[4] Let $\Delta(\mathbf{X})$ be the space of probability measures over $\mathbf{X}$; a lottery over this set is denoted by $\mu^i \in \Delta(\mathbf{X})$. We endow $\Delta(\mathbf{X})$ with the weak* topology.

An allocation in this framework is then a collection of lotteries $\{\mu^i\}_{i \in I}$, one for each agent's type, lying in the space $\Delta(\mathbf{X})^I$, endowed with the product topology.

Let $\pi^i \cdot U^i(x)$ denote a map from $\mathbf{X}$ to $\mathbb{R}$, defined by $\pi^i \cdot U^i(x) \equiv \sum_{s \in S} \pi^i(s) u^i(x(s))$, for $i \in I$. Agent $i$'s expected utility from the lottery $\mu^i$ can then be written as $\int_{\mathbf{X}} \pi^i \cdot U^i(x) d\mu^i(x)$. Similarly, let $\pi^i \cdot x$, also a map from $\mathbf{X}$ to $\mathbb{R}$, be $\pi^i \cdot x \equiv \sum_{s \in S} \pi^i(s) x(s)$.

In the analysis we will concentrate our attention on symmetric allocations, such that the same lottery is chosen by each type.[5] We therefore say that an allocation $\{\mu^i\}_{i \in I}$ is

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[4]The finite dimension, $S$, of the underlying commodity space and the finite number of agents' types imply, using Caratheodory's Theorem, that lotteries can always be taken to have a finite support, of a given cardinality. As a consequence of this, and the finiteness of aggregate resources, (under the additional condition $\lim_{v \to \infty} u^i(v)/v = 0$ for all $i$) a similar argument to the one in Kehoe, Levine and Prescott (2002) can be used to show that the restriction to lotteries over an arbitrarily large, but compact, support entails no loss of generality in our set-up.

[5]See Cres (1996) for an analysis of asymmetric equilibria of economies with markets for consumption rights, but in the set-up of finite economies and no incentive constraints.

feasible if it satisfies the following condition:

$$\sum_{i \in I} \xi^i \left( \int_{\mathbf{X}} \pi^i \cdot x d\mu^i(x) - \pi^i \cdot \omega \right) \leq 0. \tag{2.1}$$

It is incentive compatible if

$$\int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu^j(x) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu^{j'}(x), \ \forall j, j' \ \in I \tag{2.2}$$

Incentive efficient allocations maximize agents' welfare, subject to the feasibility, (2.1), and the incentive compatibility, (2.2), constraints. Thus we say:

**Definition 1.** *An allocation $\{\mu^i\}_{i \in I}$ is incentive efficient if it is feasible and incentive compatible, that is it satisfies (2.1), (2.2), and there does not exist another allocation $\{\hat{\mu}^i\}_{i \in I}$, also feasible and incentive compatible, such that:*

$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) d\hat{\mu}^i(x) \geq \int_{\mathbf{X}} \pi^i \cdot U^i(x) d\mu^i(x), \ \text{for all } i$$

*the inequality being strict for at least some $i$.*

The set of incentive efficient allocations for this class of insurance economies with adverse selection has been analyzed by Prescott and Townsend (1984) and, more recently, by Jerez (2003).

## 3. Competitive Equilibria

The trades of any agent are fully observable. As a consequence exclusive contracts, or equivalently any restriction on admissible trades, can be implemented. The notion of competitive equilibrium we consider is essentially the one introduced in Bisin and Gottardi (2006) for Rothschild-Stiglitz economies and denoted ALPT (for Arrow-Lindahl-Prescott-Townsend). The only exception is that in this paper we consider a specification in which consumption rights are produced by firms, instead of being directly produced by consumers.[6]

In the economy described any agent can trade a complete set of consumption claims contingent on each individual state. The odds of such states depend on the agents' type. Completeness requires therefore that markets and prices be differentiated according to the agents' types. Since types are not publicly observable, appropriate restrictions on trades need to be imposed to ensure that each agent self selects into the market designated for his own type. As in Prescott and Townsend (1984), we restrict the set of

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[6]We should point out that this is however by no means essential for the results obtained.

admissible trades of any agent to include only those trades that satisfy the incentive compatibility constraints. In the presence of adverse selection such constraints relate any agent's consumption to the consumption of the other agents' types, thus generating an externality. To induce agents to internalize this externality an extended set of markets is introduced, in the spirit of Arrow (1969) and Lindahl (1919). Analogously to the case of competitive markets for "pollution rights", which are accompanied by an enforcement mechanism requiring each polluter to hold an appropriate quantity of rights, at an ALPT equilibrium there are competitive markets for consumption rights and an enforcement mechanism which specifies which *rights* holding consumption rights provides the agent with.[7]

The design of the structure of markets is as follows. Consider an arbitrary agent. This agent has first to choose in which market to trade. Suppose he chooses to trade in the market designated for type $k \in I$ (thus he implicitly declares his type is $k$). He cannot trade then in any of the other markets (i.e., the markets designated for the other types). In the market he selected ($k$), the agent can trade consumption claims, contingent on every possible individual state, as well as consumption rights for each type other than $j$. To be able to purchase any amount of consumption claims, the agent must also buy an appropriate amount of rights for consumption for all types $j \neq k$. This constitutes the enforcement mechanism component of the market design. The link between holdings of claims for the agent's own consumption and of rights to the other types' consumption is designed so as to induce the agent to internalize the effects of his consumption on the incentive constraints of the other types, and explains why the agent will hold consumption rights, even though they do not directly affect his consumption level and hence his utility.

We turn now to a formal description of the market structure, the agents' choice problem, and in particular the specification of the constraints on admissible trades. Suppose an agent chooses to trade in the market designated for the $k$ types; $x_k(s; j)$ denotes then the amount purchased in this market of consumption rights for type $j \in I$, $j \neq k$ in state $s \in S$, $x_k(j) \equiv (x_k(s; j))_{s \in S} \in \mathbf{X}$. The type index appearing as a subscript indicates, here and elsewhere, the market where trades take place, while the type index appearing in the parenthesis indicates the type of agent for whom consumption rights are purchased. Similarly, $x_k(k) \in \mathbf{X}$ denotes the amount purchased in market $k$ of claims for the agent's own consumption. Every agent trading in market $k$ has then to select a vector $\mathbf{x}_k \equiv (x_k(j))_{j \in I} \in \mathbf{X}^I$. We assume that the agent cannot directly consume the amount of consumption good he is endowed with, but has to sell his entire endowment in the market.[8] The level of consumption of the agent is so equal to $x_k(k)$, the component

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[7]See Bisin Gottardi (2006, pp. 510-2) for a discussion of ways of implementing markets for consumption rights in these economies.

[8]This condition ensures that the agents' budget set has a nonempty interior at all prices. It could be relaxed however by requiring agents to sell only a positive fraction of their endowment. It should also be noticed that the condition determines a specific distribution of the endowments of rights, as discussed

of $\mathbf{x}_k$ corresponding to the type declared by the agent.

We allow agents to choose, more generally, a collection of lotteries $\mu_k$ over $\mathbf{X}^I$, i.e. an element of $\Delta(\mathbf{X})^I$; randomization is then independent across types and $\mu_k(j)$, the marginal of $\mu_k$ with respect to its $j \in I$ component, is a probability measure on $\mathbf{X}$, describing a lottery over the rights to consumption for the type $j$ agents.

The presence of a different market – designated – for each type requires that appropriate constraints are imposed on the set of admissible trades in each market, so as to ensure that every agent, even though his type is private information, at equilibrium selects to trade in the market for his own type. The constraints also specify the amount of consumption rights which must be held for any level of purchases of consumption claims. Specifically, two sets of constraints on trades are introduced.

First, the lotteries for consumption claims and consumption rights which can be traded in the market designated for any type $k \in I$ have to be mutually incentive compatible. More precisely, to be able to consume the amount given by the lottery $\mu_k(k)$, the agent must hold an amount of consumption rights, described by the lotteries $\{\mu_k(j)\}_{j \neq k}$ such that:

$$\int_{\mathbf{X}} \pi^j \cdot U^j(x) \, d\mu_k(x; j) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x) \, d\mu_k(x; k), \ \forall j \neq k, \qquad (3.1)$$

i.e., such that type $j$ agents prefer the $j$ to the $k$ component of $\mu_k$.[9] This constraint defines what is the appropriate level of rights to be held, which allows to internalize the externality arising from the incentive compatibility restrictions and to generate transfers among different types of agents. The constraint ensures, as we shall see, that at equilibrium type $j$ agents prefer to trade in the market for $j$ rather than trading in the market for any other $k \neq j$ and making the same trades as the agents who chose to trade in that market. However, it does not ensure that type $j$ agents do not prefer also any other trade in the market for $k$ which satisfies the above constraint.

To this end an additional constraint has to be imposed, requiring that an agent cannot make a trade in the market designated for type $k$ that only a type $j \neq k$ would choose in that market; such trade would in fact reveal the agent was lying when he declared his type to be $k$. More precisely, we require the set of admissible trades to be such that:

$$\begin{array}{l} \text{if } \int_{\mathbf{X}} \pi^k \cdot U^k(x) \, d\mu_k(x; k) < \int_{\mathbf{X}} \pi^k \cdot U^k(x) \, d\bar{\mu}_k(x; k), \\ \text{then } \int_{\mathbf{X}} \pi^j \cdot U^j(x) \, d\mu_k(x; k) < \int_{\mathbf{X}} \pi^j \cdot U^j(x) \, d\bar{\mu}_k(x; k), \ \forall j \neq k \end{array}, \qquad (3.2)$$

where $\bar{\mu}_k(k)$ describes the conjecture entertained by agents over the amount of consumption claims chosen by the agents who select to trade in the market for $k$. Such conjecture

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in Bisin and Gottardi (2006, p. 509).

[9] These are only a subset of the incentive compatibility constraints, described in (2.2). The constraints requiring any type $j$ to prefer the $j$ component of $\mu_k$ also to any other component $j' \neq k$ are missing. However, as we will show later, at equilibrium the missing constraints are always satisfied.

is taken as exogenously given and at equilibrium we require it to be equal to the trade actually made in the market for $k$. The constraint says that agents cannot trade in market $k$ lotteries $\mu_k(k)$ which give type $k$ agents a lower utility than $\bar{\mu}_k(k)$ but a higher utility to some other type $j \neq k$. We should stress that presence of this constraint is necessary for markets to be viable in our economy: Rustichini and Siconolfi (2008) do not impose it and find that competitive equilibria may not exist.

Prices are linear in the quantities (of contingent consumption claims and consumption rights) traded; lotteries are priced then according to their expected use (of claims and rights). The unit price in the market for $k$ of the consumption right in state $s$ for type $j$ agents is denoted by $p_k(s,j)$. The vector of the prices for the consumption rights and consumption claims in the market for $k$ is then $\mathbf{p}_k \equiv (p_k(j))_{j \in I} \in \mathbb{R}_+^{SI}$, where $p_k(j) \equiv (p_k(s,j))_{s \in S}$. The prices at which the agent sells claims to his endowment $(\omega)$ are then $\mathbf{q}_k \equiv (q_k(s))_{s \in S} \in \mathbb{R}_+^S$.[10] Define then the maps, from $\mathbf{X}$ to $\mathbb{R}$ : $p_k(j) \cdot x(j) \equiv \sum_{s \in S} p_k(s,j)x(s;j)$, $j \in I$, and $\mathbf{q}_k \cdot \boldsymbol{\omega} \equiv \sum_{s \in S} q_k(s)\omega(s)$.

Summarizing, the set of budget feasible and admissible trades in the market designated for type $k$ agents is defined by:[11]

$$
T_k(\mathbf{p}_k, \mathbf{q}_k) = \begin{cases}
\mu_k \in \Delta(\mathbf{X})^I \text{ such that:} \\
\text{(i) } \int_{\mathbf{X}} \pi^j \cdot U^j(x)\, d\mu_k(x;j) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x)\, d\mu_k(x;k), \ \forall j \neq k, \\
\text{(ii)} \quad \begin{array}{l} \text{if } \int_{\mathbf{X}} \pi^k \cdot U^k(x)\, d\mu_k(x;k) < \int_{\mathbf{X}} \pi^k \cdot U^k(x)\, d\bar{\mu}_k(x;k), \\ \text{then } \int_{\mathbf{X}} \pi^j \cdot U^j(x)\, d\mu_k(x;k) < \int_{\mathbf{X}} \pi^j \cdot U^j(x)\, d\bar{\mu}_k(x;k), \ \forall j \neq k \end{array} \\
\text{(iii) } \int_{\mathbf{X}} \sum_{j \in I} p_k(j) \cdot x(j) d\mu_k(x;j) - \mathbf{q}_k \cdot \boldsymbol{\omega} \leq 0
\end{cases}
$$

Condition (iii) is the budget constraint, while (i) and (ii) are the incentive constraints described above restricting admissible trades in the market for $k$. The corresponding sets for the other markets, $T_j(\mathbf{p}_j, \mathbf{q}_j)$, $j \in I$, are similarly defined. As we will see, conditions (i) and (ii) ensure that, at equilibrium, agents of type $j$ select to trade in the market designated for their true type $j$, for every $j \in I$.

We are now ready to write the optimization problem of an agent of generic type $i \in I$. Such agent faces one set of market for each type $j \in I$ and has to select in which market

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[10]The price paid to purchase a consumption claim in market $k$, $p_k(k)$, may differ from the price, $\mathbf{q}_k$, at which the endowment of goods is sold in that market. This reflects the fact that the enforcement mechanism in place, requiring an appropriate purchase also of consumption rights, only applies to the purchases of claims, not to the sales of the endowment.

[11]In condition (ii) the term $\bar{\mu}_k(k)$ appears, which describes the (agents' conjecture about the) trades made in market $k$. This may seem to introduce another externality in the specification of the agents' trading set. However, it is not so. We show in fact in the next section (see Lemma 1) that the only role of condition (ii) is to ensure that agents self-select into the markets designated for their own type, but does not affect equilibrium allocations in any other way. And, as discussed in Bisin and Gottardi (2006, p. 506), alternative, equivalent specifications of this additional constraint are possible, which do not require the consideration of conjectures about the level of trades made in the various markets.

he chooses to trade:

$$\max_{\{\mu_j\}_{j\in I}} \int_{\mathbf{X}} \sum_j \pi^i \cdot U^i(x) d\mu_j(x; j) \qquad (P^i_{ALPT})$$

subject to:

$$\begin{cases} \mu_j \in T_j(\mathbf{p}_j, \mathbf{q}_j) \text{ for all } j \in I, \\ \mu_k \neq \delta_0 \implies \mu_j = \delta_0 \text{ for all } j \neq k \end{cases},$$

where $\delta_0$ denotes the Dirac measure on 0.

The supply of consumption claims as well as of consumption rights[12] come from firms. The latter are produced transforming aggregates of endowment claims purchased from the consumers into consumption claims, contingent on individual states and types, using the technology provided by the Law of Large Numbers. In doing so firms act as insurance companies, as firms in Prescott and Townsend (1984). At the same time, for any amount of consumption claims which is produced a firm also produces a corresponding amount of consumption rights.

Formally, let $M_+(\mathbf{X})$ denote the set of positive measures defined on the Borel sets of $X$. [13] The term $\rho_k \in M_+(\mathbf{X})$ indicates then the amount of the lottery over claims to the endowment purchased by a firm (on a per capita basis) in the market designated for the type $k$ agents; $\nu_k(k) \in M_+(\mathbf{X})$ is the amount of the lottery over claims for consumption (also on a per capita basis) which is produced by the firm and can be sold in the market designated for type $k$, and $\nu_k(j) \in \Delta(\mathbf{X})$, for $j \neq k$, is the lottery over consumption rights for type $j$ agents which can also be sold in the market for $k$. The firms' technology is then described by:

$$\mathcal{N}^{ALPT} = \left\{ \begin{array}{c} (\rho_k, \nu_k) \in M_+(\mathbf{X}) \times M_+(\mathbf{X})^I, \ k \in I : \\ \text{(i)} \ \nu_k(x; k) = \nu_j(\frac{\xi^k}{\xi^j} x; k) \text{ for } x \in \mathbf{X}, \text{ for all } j \neq k, \\ \text{(ii)} \ \sum_{k\in I} \int_{\mathbf{X}} \pi^k \cdot x(d\nu_k(x; k) - d\rho_k(x)) \leq 0 \end{array} \right\},$$

where condition (i) prescribes the amount produced of consumption rights to be backed by a corresponding level of production of consumption claims, and (ii) requires the latter to be obtained from the transformation of an appropriate amount of endowment claims. Note that the technology is characterized by constant returns to scale[14].

The firms' problem is then the choice of a lottery $(\rho_k, \nu_k)_{k\in I} \in \mathcal{N}^{ALPT}$ so as to maximize profits:

$$\max_{(\rho_k, \nu_k)_{k\in I} \in \mathcal{N}^{ALPT}} \sum_{k\in I} \left( \int_{\mathbf{X}} \sum_{j\in I} p_k(j) \cdot x(j) d\nu_k(x; j) - \int_{\mathbf{X}} \mathbf{q}_k \cdot x d\rho_k(x) \right), \qquad (P^f_{ALPT})$$

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[12] As already noticed, in Bisin and Gottardi (2006) an alternative market mechanism is considered, in which consumers produce the rights themselves. The specification in the present paper is more in accordance with the standard analysis of Lindahl economies in the public finance literature.

[13] This set obviously includes $\Delta(\mathbf{X})$.

[14] We can avoid, as a consequence, to specify the number of existing of firms and proceed, without loss of generality, as if a single, representative firm existed.

where $\mathbf{q}_k \cdot x \equiv \sum_{s \in S} q_k(s)\, x(s)$, taking prices $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$ as given.

**Definition 2.** *An ALPT equilibrium is a collection of prices* $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$, *consumption lotteries for each consumer's type* $\{\mu_k^i\}_{i,k \in I}$, *production lotteries* $(\rho_k, \nu_k)_{k \in I}$ *and conjectures* $(\bar{\mu}_k(k))_{k \in I}$ *such that:*

**(i)** *for each* $i$, $\{\mu_k^i\}_{k \in I}$ *is a solution of type* $i$'s *maximization problem* $(P_{ALPT}^i)$, *given prices* $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$ *and* $(\bar{\mu}_k(k))_{k \in I}$;

**(ii)** $(\rho_k, \nu_k)_{k \in I}$ *solves the firms' profit maximization problem* $(P_{ALPT}^f)$ *at the prices* $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$;

**(iii)** *every agent chooses to trade in the market designated for his true type* $(\mu_j^i = \delta_0$ *for all* $j \neq i)$ *and the conjecture about the lottery over consumption claims traded in each market,* $(\bar{\mu}_k(k))$, *is consistent with the actual choice made by agents who traded in that market:*

$$\bar{\mu}_k(k) = \mu_k^k(k), \text{ for all } k \in I.$$

**(iv)** *markets for consumption rights and consumption claims clear:*

$$\mu_i^i(\xi^i x; j) = \nu_i(x; j) \text{ for } x \in \mathbf{X}, \text{ for all } i, j;$$

**(v)** *markets for endowment claims clear:*

$$\int_{\mathbf{X}} x d\rho_i(x) \leq \xi^i \omega, \qquad \text{for all } i;$$

From the market clearing condition for consumption rights ((iv) above), the fact that agents self select into the market for their own type (property (iii)) and the specification of the firms' technology in $(\mathcal{N}^{ALPT})$, it follows that, at an ALPT equilibrium, we have:

$$\mu_j^j = \mu_i^i, \quad \text{for any } i, j \in I, \tag{3.3}$$

or the lotteries maximizing agents' utilities have to be the same in every market. Moreover, such lotteries have to satisfy the incentive compatibility constraints (3.1) for all $i$, so that type $i$ agents prefer the $i$-th component of such lottery $\mu_i^i(i)$ while type $j$ prefer the $j$-th component $\mu_i^i(j)$, for all $i, j$. As a consequence, the equilibrium allocation specified by $\mu_i^i$ is mutually incentive compatible, i.e. satisfies (2.2).

## 4. Existence and Efficiency of Equilibria

We establish first an important preliminary result in our analysis of ALPT equilibria. Consider the following specification of the choice problem of type $i$ agents:

$$\max_{\mu_i \in T_i^A(\mathbf{p}_i, \mathbf{q}_i)} \int_{\mathbf{X}} \pi^i \cdot U^i(x) d\mu_i(x; i), \qquad (AP_{ALPT}^i)$$

where:

$$T_i^A(\mathbf{p}_i, \mathbf{q}_i) = \left\{ \begin{array}{c} \mu_i \in \Delta\left(\mathbf{X}\right)^I \text{ such that:} \\ \text{(i) } \int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu_i(x; j) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu_i(x; i), \forall j \neq i, \\ \text{(ii) } \int_{\mathbf{X}^I} \mathbf{p}_i \cdot \mathbf{x} d\mu_i(x) - \mathbf{q}_i \cdot \boldsymbol{\omega} \leq 0 \end{array} \right.$$

This differs from the one considered in the definition of ALPT equilibria, $(P_{ALPT}^i)$, in two respects: a) agents have to trade in the market designated for their own type; b) the set of admissible trades in such market is only defined by what were conditions (i) and (iii) in the specification of $T_k(\mathbf{p}_k, \mathbf{q}_k)$. Evidently, a) makes this a purely artificial construction since, when the agents' type is only privately observable, we cannot guarantee that they sort themselves correctly into the market designated for their own type.

However, our next Lemma shows the following: if we consider this alternative formulation of the agents' choice problem, but leave otherwise unchanged the notion of equilibrium as in Definition 2, any set of equilibrium prices and allocations we find also constitutes an ALPT equilibrium, that is an equilibrium when agents are free to choose in which market to operate but the additional constraint (3.2) restricts the set of admissible trades in each market, and viceversa. In other words, (3.2) guarantees the correct sorting of agents into the market designated for their own type, but does so without effectively restricting the lotteries that agents of type $i$ would trade in market $i$. It only restricts the lotteries that agents of type $j \neq i$ would trade in market $i$ (if they selected to trade in such market, which never happens in equilibrium).[15]

Formally, we say that a collection of prices $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$, consumption and production lotteries $\{\mu_i^i\}_{i \in I}$, $(\rho_k, \nu_k)_{k \in I}$, such that conditions (ii),(iv) and (v) of Definition 2 hold and, for all $i$, $\mu_i^i$ solves problem $(AP_{ALPT}^i)$ at the prices $(\mathbf{p}_i, \mathbf{q}_i)$ defines an *auxiliary equilibrium*. We then have:

**Lemma 1.** *The set of ALPT equilibria and auxiliary equilibria coincide.*

From this Lemma it follows that, to establish the existence and characterize the properties of ALPT equilibria, it suffices to consider a simpler version of the agents'

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[15]Of course this does not mean that (3.2) is redundant in our definition of equilibrium. We have already pointed out in fact that without the constraint equilibria may not exist, as in Rustichini and Siconolfi (2008).

optimization problem given by $(AP^i_{ALPT})$, where constraint (3.2) is no longer imposed and type $i$ agents are required to trade in the market for $i$.

**Proof.** Let $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$, $\{\mu^i_k\}_{i,k \in I}$, $(\rho_k, \nu_k)_{k \in I}$, $(\bar{\mu}_k(k))_{k \in I}$ be an ALPT equilibrium. At such equilibrium, from condition (iii) of Definition 2 it follows that agents of type $i$ choose to trade in the market for type $i$ agents, for all $i$, that is $\mu^i_j = \delta_0$ for all $j \neq i$. For the ALPT equilibrium allocation to constitute also an auxiliary equilibrium it suffices then to show that $\mu^i_i$ is a solution of problem $(AP^i_{ALPT})$ at prices $(\mathbf{p}_i, \mathbf{q}_i)$, or equivalently that we cannot find another lottery $\tilde{\mu} \in \Delta(\mathbf{X})^I$, satisfying the constraints in $(AP^i_{ALPT})$, such that $\int_{\mathbf{X}} \pi^i \cdot U^i(x) d\tilde{\mu}(x; i) > \int_{\mathbf{X}} \pi^i \cdot U^i(x) d\mu^i_i(x; i)$. But this is immediate, since $\tilde{\mu}$ obviously satisfies (3.2), so that $\tilde{\mu} \in T_i(\mathbf{p}_i, \mathbf{q}_i)$, which would contradict the optimality of $(\mu^i_i)$.

Next, suppose $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$, $\{\mu^i_i\}_{i \in I}$, $(\rho_k, \nu_k)_{k \in I}$ is an auxiliary equilibrium. We will show that these values, together with $\{\mu^i_j = \delta_0 \,\forall\, j \neq i\}_{i \in I}$ and $\bar{\mu}_k(k) = \mu^k_k(k)$ for all $k \in I$, also constitute an ALPT equilibrium. This is true if, for all $i \in I$, $\{\mu^i_i; \mu^i_j = \delta_0 \,\forall\, j \neq i\}$ is a solution of problem $(P^i_{ALPT})$. Evidently, type $i$ agents cannot find an improving trade in the market for $i$ (or $\mu^i_i$ would not be a solution of $(AP^i_{ALPT})$). Thus the issue is whether $i$ could improve by claiming to be of a different type $j \neq i$. Condition (ii) in the specification of $T_j(\mathbf{p}_j, \mathbf{q}_j)$ implies that the best type $i$ can do in the market for $j$, when $\bar{\mu}_j(j) = \mu^j_j(j)$ (i.e., when $\bar{\mu}_j(j)$ is the best type $j$ agents can do in such market at the current prices), is indeed to choose $\bar{\mu}_j(j)$. Since, by essentially the same argument as the one used for ALPT equilibria at the end of the previous section we can show that at an auxiliary equilibrium $\mu^i_i = \mu^j_j$ and the allocation specified by such lottery is mutually incentive compatible, it cannot be that type $i$ prefers its $j$ component $\mu^j_j(j)$ to the $i$ component $\mu^i_i(i)$. This establishes the claim. ∎

We can now prove the main result of the paper:

**Theorem 4.1.** *Under Assumption 1, ALPT equilibria exist.*

**Proof.** In virtue of Lemma 1, it suffices to show the existence of an auxiliary equilibrium. The proof is organized as follows. We consider first the problem of an agent of type $i \in I$, who has to choose a lottery $\mu \in \Delta(\mathbf{X})^I$ subject to the incentive compatibility constraints and the constraint that the expected value of the lottery (i.e., the expected amount obtained of consumption claims in each state and of consumption rights also in each state for every other consumer's type) is fixed at some level $\bar{\mathbf{x}}^i = (\bar{x}^i(j))_{j \in I} \in \mathbf{X}^I$. A key step of the argument is then to show that, whenever this vector is the same for agents of types $i$ and $j$, i.e. $\bar{\mathbf{x}}^i = \bar{\mathbf{x}}^j$, the lotteries chosen are also the same: $\mu^i = \mu^j$, for all $i, j \in I$. On this basis, we can construct the indirect utility function of an agent of type $i$ as a function of $\bar{\mathbf{x}}^i \in \mathbf{X}^I$, for all $i \in I$, and establish market clearing using the simpler, finite-dimensional space $\mathbf{X}^I$ as commodity space. To this end, we show that

such indirect utility function exhibits all standard properties of agents' utility functions. Hence the demand obtained by choosing $\bar{x}^i \in \mathbf{X}^I$ so as to maximize the indirect utility function of agent $i$, subject to the budget constraint, is well behaved. The existence of prices such that markets clear follows then by a standard argument.

The idea of separating the agent's choice of a lottery from the determination of its expected value together with that of equilibrium prices has been earlier utilized by Kehoe, Levine and Prescott (2002) in economies with hidden information rather than adverse selection[16]. They refer to the individual problem of choosing the lottery as "the stand-in consumer" problem.

**The stand-in consumer problem.** The problem for any agent $i \in I$ is:

$$\max_{\mu \in \Delta(\mathbf{X})^I} \int_{\mathbf{X}} \pi^i \cdot U^i(x)\, d\mu(x; i) \qquad (P^i_{STIN})$$

subject to

$$\int_{\mathbf{X}} x\, d\mu(x; j) \leq \bar{x}^i(j), \text{ for all } j \in I \qquad (4.1)$$

$$\int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu(x; j) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu(x; j'), \ \forall j, j' \in I \qquad (4.2)$$

Under our assumptions the stand-in consumer problem is a convex problem, on a compact set. Therefore the set of solutions of this problem is non-empty; moreover, it is a convex set. Let $\mu^i(\bar{\mathbf{x}}^i)$ be the set of its solutions when the expected amount of consumption claims and rights (for each type in each state) is given by $\bar{\mathbf{x}}^i$.

**Lemma 2.** *The solution $\mu^i(.)$ of the stand-in consumer problem, for any $i \in I$, has the property that:*

$$\mu^i(\bar{\mathbf{x}}^i) = \mu^j(\bar{\mathbf{x}}^j) \text{ whenever } \bar{\mathbf{x}}^i = \bar{\mathbf{x}}^j$$

**Proof.** Consider the following variant of the above stand-in problem, where the constraint set is unchanged, but the objective function is replaced by a weighted average of the utilities of all agents' types, with weights $\theta \in \mathbb{R}^I_+$ such that $\sum_{i \in I} \theta_i = 1$ :

$$\max_{\mu \in \Delta(X)^I} \sum_{i \in I} \theta^i \int_{\mathbf{X}} \pi^i \cdot U^i(x)\, d\mu(x; i) \qquad (P^*_{STIN})$$

subject to

$$\int_{\mathbf{X}} x\, d\mu(x; j) \leq \bar{x}(j), \forall j \in I \qquad (4.3)$$

$$\int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu(x; j) \geq \int_{\mathbf{X}} \pi^j \cdot U^j(x) d\mu(x; j'), \ \forall j, j' \in I \qquad (4.4)$$

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[16]Hence with no markets for consumption rights.

Let $\mu^*(\bar{\mathbf{x}};\theta) \in \Delta(X)^I$ denote the solution of this problem, as a function both of $\bar{\mathbf{x}} = (\bar{x}(j))_{j \in I}$ and of $\theta$.

The auxiliary problem has the property that, when $\theta_i = 1$ for some $i \in I$ and $\bar{\mathbf{x}} = \bar{\mathbf{x}}^i$, it coincides with the stand-in consumer problem of type $i$ agents; hence, for such value of $\theta$, $\mu^*(\bar{\mathbf{x}}^i;\theta)$ constitutes a solution of the stand-in consumer problem $(P^i_{STIN})$ of the type $i$ agents. Therefore, the statement of the Lemma follows if we can show that, for all $\bar{\mathbf{x}} \in \mathbf{X}^I$:

$$\mu^*(\bar{\mathbf{x}};\theta) = \mu^*(\bar{\mathbf{x}};\theta'), \text{ for all } \theta, \theta' \in \mathbb{R}^I_+ \text{ such that } \sum_{i \in I} \theta_i = \sum_{i \in I} \theta'_i = 1 \qquad (4.5)$$

We show first that (4.5) holds for all $\theta \gg 0$. Suppose not, i.e. there are distinct lotteries $\widehat{\mu} \neq \mu$ which solve the problem respectively for $\hat{\theta}$ and $\theta$, for some given $\bar{\mathbf{x}}$. As a consequence, for $i$ belonging to a non-empty subset $\hat{I}$ of $I$, we have

$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widehat{\mu}(x;i) \geq \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\mu(x;i)$$

and for $i$ in the complement subset $I \backslash \hat{I}$ of $I$, also non-empty:

$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\mu(x;i) > \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widehat{\mu}(x;i)$$

Consider then the collection of lotteries $\widetilde{\mu}$ constructed as the product of $(\widehat{\mu}(i))_{i \in \hat{I}}$ and $(\mu(i))_{i \in I \backslash \hat{I}}$. Trivially, $\widetilde{\mu}$ satisfies the resource feasibility constraints (4.3). The incentive constraints (4.4) are then also satisfied. In fact, for any $i \in \hat{I}$ we have:

$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widetilde{\mu}(x;i) = \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widehat{\mu}(x;i) \geq$$
$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\mu(x;i) \geq \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\mu(x;j) = \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widetilde{\mu}(x;j), \; j \in I \backslash \hat{I}$$

where the inequality in the second line follows from the fact that $(\mu(i))_{i \in I}$ satisfies (4.4), and the equalities from the above construction of $\widetilde{\mu}$. Furthermore:

$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widetilde{\mu}(x;i) = \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widehat{\mu}(x;i) \geq$$
$$\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widehat{\mu}(x;j) = \int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\widetilde{\mu}(x;j), \; j \in \hat{I}.$$

A similar argument can then be used to establish the analogous property for all $i \in I \backslash \hat{I}$.

Hence $\widetilde{\mu}$ is also an admissible solution of the problem $(P^*_{STIN})$ for the given value of $\theta$ and yields a higher value of the objective function, a contradiction.

A straightforward continuity argument then implies that (4.5) also holds for all $\theta > 0$, thus completing the proof. ∎

Let $V^i(\bar{\mathbf{x}}^i)$ denote the value of $\int_{\mathbf{X}} \pi^i \cdot U^i(x) \, d\mu(x; i)$ at $\mu^i(\bar{\mathbf{x}}^i)$, i.e. the solution of the stand-in consumer problem $(P^i_{STIN})$ for a given level of resources $\bar{\mathbf{x}}^i$ (specifying the expected value of the lottery for each type in every state). This defines a map from $\mathbf{X}^I$ to $\mathbb{R}$, representing $i$'s indirect utility.

We study next the properties of $V^i(\bar{\mathbf{x}}^i)$.

**Lemma 3.** *The indirect utility function $V^i(\bar{\mathbf{x}}^i)$ is well-defined and continuous for all $\bar{\mathbf{x}}^i \in \mathbf{X}^I$, weakly monotonic, weakly quasi-concave, and satisfies local non satiation.*

**Proof.** Continuity follows from the Maximum theorem (Berge (1963)). It is then straightforward to verify that $V^i(\bar{\mathbf{x}}^i)$ is weakly monotonic.

We now show that $V^i(\bar{\mathbf{x}}^i)$ is weakly quasi-concave. Let $\bar{\mathbf{x}}^i$ and $\bar{\mathbf{x}}'^i$ be two vectors such that $V^i(\bar{\mathbf{x}}^i) \geq V^i(\bar{\mathbf{x}}'^i)$ and $\mu^i$ and $\mu'^i$ the collections of lotteries supporting, respectively, $V^i(\bar{\mathbf{x}}^i)$ and $V^i(\bar{\mathbf{x}}'^i)$ (i.e., $\mu^i = \mu^i(\bar{\mathbf{x}}^i), \mu'^i = \mu^i(\bar{\mathbf{x}}'^i)$). Consider then, for $\alpha \in [0, 1]$, the vector $\bar{\mathbf{x}}''^i \equiv \alpha \bar{\mathbf{x}}'^i + (1 - \alpha)\bar{\mathbf{x}}^i$ and the lotteries $\mu''^i \equiv \alpha \mu'^i + (1 - \alpha)\mu^i$. The lotteries $\mu''^i$ clearly satisfy the constraints (4.1-4.2) of problem $(P^i_{STIN})$ when the level of resources is given by $\bar{\mathbf{x}}''^i$ (this is immediate, given the linearity of such constraints). As a consequence, we have:

$$V^i(\bar{\mathbf{x}}''^i) \geq \alpha V^i\left(\bar{\mathbf{x}}'^i\right) + (1 - \alpha)V^i\left(\bar{\mathbf{x}}^i\right) \geq V^i(\bar{\mathbf{x}}'^i),$$

which proves the weak quasi-concavity of $V^i(\bar{\mathbf{x}}^i)$.

It remains to show that $V^i(\bar{\mathbf{x}}^i)$ also satisfies the local non-satiation property, at any point $\bar{\mathbf{x}}^i$. Pick some vector $x^M \in \mathbf{X}$ with all its elements near the upper bound $M$ and let $\delta_{x^M}$ denote the Dirac measure on $x^M$. Consider then the lotteries:

$$\mu^i(j; \bar{\mathbf{x}}^i, \alpha) \equiv (1 - \alpha)\mu^i(j; \bar{\mathbf{x}}^i) + \alpha \delta_{x^M}, \qquad j \in I$$

for $\alpha \in (0, 1]$. For any arbitrarily small neighborhood of $\bar{\mathbf{x}}^i$ we can find a set of values of $\alpha$, sufficiently close to zero, such that the expected values of the resources used by the associated lotteries $(\mu^i(j; \bar{\mathbf{x}}^i, \alpha))_{j \in I}$ lie within this neighborhood. It is then immediate to verify that such lotteries also satisfy the incentive compatibility constraints (4.2). By construction (in particular, by the choice of $x^M$), $i$'s utility is strictly higher: $\int_{\mathbf{X}} \pi_i \cdot U^i(x) \, d\mu^i(i; \bar{\mathbf{x}}^i, \alpha) > \int_{\mathbf{X}} \pi_i \cdot U^i(x) \, d\mu^i(i; \bar{\mathbf{x}}^i)$, thus completing the argument. ∎

On the basis of the properties showed in Lemmas 2 and 3, the existence of ALPT equilibria can be established by working with the finite dimensional set $\mathbf{X}^I$ as commodity space.

The demand of consumers of type $i \in I$ is obtained by maximizing their indirect utility function $V^i(\bar{\mathbf{x}}^i)$, defined over this space, subject to the budget constraint:

$$\max_{\bar{\mathbf{x}}^i \in \mathbf{X}^I} V^i(\bar{\mathbf{x}}^i) \qquad\qquad (P^i_{red})$$

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subject to

$$\mathbf{p}_i \cdot \bar{\mathbf{x}}^i - \mathbf{q}_i \cdot \boldsymbol{\omega} \leq \mathbf{0}$$

The problem of the firm remains convex even if we restrict its choices to lie in the underlying, finite dimensional commodity space $\mathbf{X}^I$ (or to choose degenerate lotteries). In this case, the firm's technology can be described more simply by the set:

$$N^{ALPT} = \left\{ \begin{array}{l} (r_k, \mathbf{n}_k) \in \mathbf{X} \times \mathbf{X}^I, \ k \in I : \\ n_k(k) = \frac{\xi^k}{\xi^j} \, n_k(j) \text{ for all } k, j \\ \sum_{k \in I} \pi^k \cdot (n_k(k) - r_k) \leq 0 \end{array} \right\}$$

It can easily be verified that $N^{ALPT}$ is (still characterized by constant returns to scale,) closed, convex, includes the origin and satisfies (a weaker form of) free disposal property.

The problem of the firm can so be rewritten as follows:

$$\max_{(r_k, \mathbf{n}_k)_{k \in I} \in N^{ALPT}} \sum_{k \in I} \mathbf{p}_k \cdot \mathbf{n}_k - \sum_{k \in I} \mathbf{q}_k \cdot r_k \qquad (P_{red}^f)$$

The market clearing conditions become:

$$\xi^i \bar{\mathbf{x}}^i = \mathbf{n}_i, \ \forall i \in I$$

$$r_i \leq \xi^i \omega, \qquad i \in I$$

Existence of a quasi equilibrium (more precisely, of a set of prices $(\mathbf{p}_k, \mathbf{q}_k)_{k \in I}$, firms' choices $(r_k, \mathbf{n}_k)_{k \in I}$ solving $\left( P_{red}^f \right)$ and consumers' choices $(\bar{\mathbf{x}}^i)_{i \in I}$ solving the expenditure minimization problem associated to $(P_{red}^i))$ follows then by an application of known results (e.g. Proposition 17.BB.2 in Mas-Colell, Green and Whinston (1995)). Moreover, we have $\mathbf{q}_i = \pi^i$, so that the income of every agent is strictly positive, thus ensuring that the one found is actually a competitive equilibrium. This completes the proof of the Theorem. ∎

The welfare properties of ALPT equilibrium allocations are then also immediately established. Since, as shown in Lemma 3, agents' preferences satisfy the local non-satiation property, the validity of the first welfare theorem follows by a standard argument:

**Theorem 4.2.** *Any ALPT equilibrium allocation* $(\mu_i^i(i))_{i \in I}$ *is incentive efficient.*

**Proof.** The proof is quite standard. Suppose not, i.e. there exists a feasible, incentive compatible allocation $(\widehat{\mu}^i)_{i \in I}$ which Pareto dominates the equilibrium allocation $(\mu_i^i(i))_{i \in I}$. Then, the following constitutes an admissible choice for agent $i \in I$ (more precisely, satisfies conditions (i)-(ii) defining the set $T_i(\mathbf{p}_i, \mathbf{q}_i)$ of admissible trades in

market $i$, for $\bar{\mu}_k(k) = \mu_k^k(k)$ for all $k \in I$): $\tilde{\mu}_i(i) = \widehat{\mu}^i, \tilde{\mu}_i(j) = \widehat{\mu}^j$, for $j \neq i$. Given the fact that, as argued in the Proof of Lemma 3, local nonsatiation holds, we must have:

$$\int_{\mathbf{X}} \sum_{j \in I} p_i(j) \cdot x(j) d\tilde{\mu}_i(x; j) \geqslant \int_{\mathbf{X}} \sum_{j \in I} p_i(j) \cdot x(j) d\mu_i^i(x; j).$$

Iterating the same argument for all other types $j \neq i$, we obtain that an analogous inequality holds for all other $j \in I$, with a strict inequality for at least some $k \in I$. We can then sum these inequalities across $i$, and use the fact that $\tilde{\mu}_i$, as constructed above, is invariant with respect to $i$, and that the same is true – as we showed in the previous section – for the equilibrium lotteries $\mu_i^i$, to get:

$$\int_{\mathbf{X}} \sum_{j \in I} \left( \sum_i p_i(j) \right) \cdot x_i(j) d\tilde{\mu}_i(x; j) > \int_{\mathbf{X}} \sum_{j \in I} \left( \sum_i p_i(j) \right) \cdot x_i(j) d\mu_i^i(x; j).$$

Since $\left( \hat{\mu}^i \right)_{i \in I}$ is a feasible allocation, the bundle $(\nu_i(x; j) = \hat{\mu}^j \left( \xi^i x \right)$ for all $x \in \mathbf{X}$, $\rho_i = \delta_{\xi^i \omega}$, for all $i, j$) lies in the firms' production possibilities' set $\mathcal{N}^{ALPT}$ and, by the previous inequality, yields strictly higher profits than the production plan chosen by the firms at equilibrium, $(\nu_i(x; j) = \mu_j^j \left( \xi^i x; j \right)$ for all $x \in \mathbf{X}$, $\rho_i = \delta_{\xi^i \omega}$, for all $i, j$), thus a contradiction. ∎

## 5. Conclusions and Generalizations

In this paper we have shown the existence and efficiency of Walrasian equilibria for the general class of insurance economies with adverse selection considered by Prescott and Townsend (1984). The notion of equilibrium we use is the one introduced in our previous work, Bisin and Gottardi (2006), and is characterized by the presence of competitive markets for consumption rights, in the spirit of the notion of Arrow-Lindahl equilibria for economies with externalities. The fact that the set of admissible trades is restricted by the incentive compatibility constraints implies that this set is generally non-convex. As we argued, in the presence of non-convexities, the existence of Arrow-Lindahl equilibria is problematic, since equilibrium allocations have to satisfy a consistency condition (see (3.3) above), whereby the amount of consumption rights purchased has to be exactly the same for all agents' types. Thus the extension of the commodity space to include lotteries is not enough in this case to overcome the non-convexity and ensure existence of equilibria for large economies. Guaranteeing the validity of such consistency condition is the main difficulty and achievement of the existence proof presented here.

Our existence result cannot unfortunately be directly generalized to show the existence of Arrow-Lindahl equilibria for general economies with externalities, or public

goods, in the presence of non-convexities. The key step in the proof allowing us to establish the consistency condition, Lemma 2, uses in fact the specific properties of the externality (in consumption) generated by the presence of the incentive compatibility constraints in the specification of admissible trades to show that the lotteries chosen by different types are exactly the same when the average use of resources is the same. Such result does not extend to other, more general forms of externalities in consumption with non-convexities.

On the other hand, our methods can be applied to deal with informational issues in general economies with public goods or consumption externalities, whenever the convexity of the consumption and production sets is guaranteed. The analysis of Roberts (1979) reveals in fact that the presence of markets for consumption rights indexed by the agents' type may not be sustainable (or equivalently, Arrow-Lindahl equilibrium allocations may fail to be incentive compatible). In this case a fairly immediate reformulation of our analysis shows that such differentiation of markets can be sustained by imposing the appropriate incentive compatibility restrictions on the set of admissible trades, together with a condition analogous to (3.2) above (incentive compatibility of market differentiation). Such restrictions may generate non-convexities as we saw, but our argument can again be used to deal with them, and show the existence and incentive efficiency of competitive equilibrium allocations.

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