Appendix A: Additional material

Characterization of the firms' optimal capital structure conditions

Let I^e (resp. I^d) denote the set of shareholders (resp. bondholders) of a firm and consider for simplicity the case where capital is the only input, that is the technology is given by $f(k,s)$.

Proposition A. 1 *If the optimal production and financing decisions of a firm are obtained* [56] *at a level B such that bonds are risk free, that is, $f(k;s) \geq B$ with probability 1, then all equity holders are also bond holders (while the reverse may not be true: $I^e \subseteq I^d$):*

$$\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = p = \max_i \mathbb{E} MRS^i(c^i(s)) \tag{38}$$

and

$$\max_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) f_k(s)\right] = \min_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) f_k(s)\right] = 1; \tag{39}$$

In the situation described above all shareholders value equally the effect on the payoff of equity of an infinitesimal increase in the investment level k, and such value is always equal to the marginal cost of the investment.

Proof of Proposition A. 1 *Note first that*

$$q(k, B + dB) = \max_i \mathbb{E} MRS^i(c^i(s)) \ \left[f(k;s) - B - dB\right].$$

Since for all $i \notin I^e$, $\mathbb{E} MRS^i(c^i(s))\ [f(k;s) - B] < q(k,B)$, the max in the above expression is attained for some $i \in I^e$ and hence

$$q(k, B + dB) = q(k,B) + \max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \ [-dB].$$

The right and left derivative of $q(k,B)$ with respect to B are then given by:

$$\frac{\partial q}{\partial B_+} = -\min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \ ; \quad \frac{\partial q}{\partial B_-} = -\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \tag{40}$$

and may differ. Similarly the derivatives with respect to k are:

$$\frac{\partial q}{\partial k_+} = \max_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right]; \ \frac{\partial q}{\partial k_-} = \min_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right] \tag{41}$$

[56] We focus here on the conditions concerning the investment level k and capital structure B, ignoring those regarding ϕ, which are straightforward.

where f_k denotes the derivative of f with respect to k.

The first order conditions when $f(k,\phi,m;s) \geq B$ with probability 1 are:

$$\begin{aligned} \frac{\partial V}{\partial B_+} &= \frac{\partial q}{\partial B_+} + p \leq 0, \ \frac{\partial V}{\partial k_+} = \frac{\partial q}{\partial k_+} - 1 \leq 0, \\ \frac{\partial V}{\partial B_-} &= \frac{\partial q}{\partial B_-} + p \geq 0, \ \frac{\partial V}{\partial k_-} = \frac{\partial q}{\partial k_-} - 1 \geq 0; \end{aligned} \tag{42}$$

Since (40) implies that $\frac{\partial q}{\partial B_+} \geq \frac{\partial q}{\partial B_-}$, the above conditions (with respect to B) are equivalent to:

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p = \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p = 0,$$

that is:

$$\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = p = \max_i \mathbb{E} MRS^i(c^i(s))$$

or (38) holds. Similarly, from (41) we see that $\frac{\partial q}{\partial k_+} \geq \frac{\partial q}{\partial k_-}$, the above conditions (with respect to k) are equivalent to:

$$\frac{\partial q}{\partial k_+} - 1 = \frac{\partial q}{\partial k_-} - 1 = 0,$$

that is,

$$\max_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right] = \min_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right] = 1$$

or (39) holds, thus completing the proof of the proposition. ■

We study next the case where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms' decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to B, let S^{nd} denote the collection of states in $t = 1$ for which $f(k;s) \geq B$ and by $\underline{s}^{nd}$ the lowest state in S^{nd}, that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote S^d the collection of states in $t = 1$ for which $f(k;s) < B$, i.e. the firm (partially) defaults on its debt.

Proposition A. 2 *If the optimal production and financing decisions of a firm are obtained at a level B such that bonds are risk free, the optimal investment and debt levels obtain either* *at an interior solution, where $f(k;\underline{s}^{nd})>B$, with:*

$$p=\min_{i\in I^d}\mathbb{E}\left(MRS^i(c^i(s))\left[\frac{f(k;s)}{B}\right]\mid s\in S^d\right)\Pr\{s\in S^d\}+$$
$$\min_{i\in I^e}\mathbb{E}(MRS^i(c^i(s))\big|\, s\in S^{nd})\Pr\{s\in S^{nd}\}= \tag{43}$$
$$=\max_{i\in I^d}\mathbb{E}\left(MRS^i(c^i(s))\left[\frac{f(k;s)}{B}\right]\mid s\in S^d\right)\Pr\{s\in S^d\}+$$
$$\max_{i\in I^e}\mathbb{E}\left(MRS^i(c^i(s))\big|\, s\in S^{nd}\right)\Pr\{s\in S^{nd}\}.$$

and

$$1=\max_{i\in I^e}\mathbb{E}\left\{MRS^i(c^i(s))f_k(k,s)\big|\, s\in S^{nd}\right\}\Pr\{s\in S^{nd}\}+ \tag{44}$$
$$\max_{i\in I^d}\mathbb{E}(MRS^i(c^i(s))f_k(k;s)\big|\, s\in S^d)\Pr\{s\in S^d\}$$
$$=\quad\min_{i\in I^e}\mathbb{E}\left\{MRS^i(c^i(s_1))f_k(k,s)\big|\, s\in S^{nd}\right\}\Pr\{s\in S^{nd}\}+$$
$$\min_{i\in I^d}\mathbb{E}(MRS^i(c^i(s))f_k(k;s)\big|\, s\in S^d)\Pr\{s\in S^d\}$$

or at a corner solution, $f(k;\underline{s}^{nd})=B$.

Proof of Proposition A. 2 *We first proceed to characterize the conditions for corner solutions.*

Claim 1 *The conditions for an optimum at a corner, $f(k;\underline{s}_1^{nd})=B$, are:*

$$\min_{i\in I^e}E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+ \tag{45}$$
$$\min_{i\in I^d}E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg|\, s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}\geq p\geq$$
$$\geq\max_{i\in I^e}E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\max_{i\in I^d}E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$

$$\min_{i\in I^e}E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+ \tag{46}$$
$$\min_{i\in I^d}E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}\geq 1\geq$$
$$\geq\max_{i\in I^e}E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\max_{i\in I^d}E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$

$$
\begin{aligned}
&1 - \max_{i \in I^e} E_{s_0} \left\{ MRS^i(s_1) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \qquad (47)\\
&\max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
&\Big[- \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
&- \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) = \\
&\Big[- \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
&- \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) = \\
&1 - \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
&- \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

Proof of Claim 1 *Note first that, in this case,* $f(k; \underline{s}_1^{nd}) = B$*. Denote by* $S_1^{nd\prime} \subset S_1^{nd}$ *the collection of states in* $t = 1$ *for which the firm does not default, after marginal deviations* $dB > 0$ *and/or* $dk < 0$ *(and similarly* $S^{d\prime} \supset S^d$*). Evidently, for marginal deviations* $dB > 0$ *and/or* $dk < 0$ *the collection of such states is still given by* S_1^{nd}*.*

The partials of the price maps wrt to B *are*[57]

$$
\begin{aligned}
\frac{\partial q}{\partial B_+} &= -\min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd\prime} \right\} \Pr\{s_1 \in S^{nd\prime}\} \\
\frac{\partial q}{\partial B_-} &= -\max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial p}{\partial B_+} &= -\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \bigg| \, s_1 \in S^{d\prime}) \Pr\{s_1 \in S^{d\prime}\} \\
\frac{\partial p}{\partial B_-} &= -\max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

Analogously, the partials wrt to k *are*[58]

$$
\begin{aligned}
\frac{\partial q}{\partial k_+} &= \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
\frac{\partial q}{\partial k_-} &= \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd\prime} \right\} \Pr\{s_1 \in S^{nd\prime}\}
\end{aligned}
$$

[57] Obviously, if S^{nd} is a singleton, the right derivative is equal to 0.

[58] Obviously, if $S^{nd} = \{s_1\}$ - is a singleton - the left derivative is equal to 0.

and

$$\frac{\partial p}{\partial k_+} = \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}$$
$$\frac{\partial p}{\partial k_-} = \min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}$$

So, if $f(k;\underline{s}_1^{nd})=B$, *the FOCs wrt* B *are:*

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+}B + p\right) =$$
$$-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+$$
$$\left(-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\Big| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}B + p\right)\leq 0$$
$$\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-}B + p\right) =$$
$$-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\left(-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\Big| s_1\in S^{d})\Pr\{s_1\in S^{d}\}B + p\right)\geq 0$$

which implies (45). Finally, the FOCs wrt k *are:*

$$\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+}B\right) =$$
$$-1+\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\left(\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^{d})\Pr\{s_1\in S^{d}\}B\right)\leq 0$$
$$\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-}B\right) =$$
$$-1+\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+$$
$$\left(\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}B\right)\geq 0$$

which implies (46). Since now expectations in the terms on the two sides of the inequality are taken over different sets, such condition is a little harder to interpret. In particular we can no longer say that all equity holders have the same valuation for the marginal productivity of capital in the no default states. Rather the condition imposes some relationship between the difference among equity holders and bond holders' valuation for the marginal productivity of capital in the two situations (S^d *and* $S^{d\prime}$*).*

We also have to check in this case the optimality of k, B wrt joint deviations of B and k. As before, without loss of generality, we can restrict our attention to changes of B and k such that $f(k; \underline{s}_1^{nd}) = B$ keeps holding (the set of states for which default occurs does not change).

$$\frac{\partial V}{\partial B_+} dB + \frac{\partial V}{\partial k_+} dk = \left[\frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+} B + p \right) \right] dB + \left[-1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+} B \right) \right] dk \leq 0,$$

for $dB = f_k(\underline{s}_1^{nd}) dk > 0$; also,

$$\frac{\partial V}{\partial B_-} dB + \frac{\partial V}{\partial k_-} dk = \left[\frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-} B + p \right) \right] dB + \left[-1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-} B \right) \right] dk \geq 0$$

for $dB = f_k(\underline{s}_1^{nd}) dk < 0$. Substituting the expressions for the partials obtained above, we get

$$\Big[- \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}$$
$$- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] |s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd})$$
$$- 1 + \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$
$$\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \leq 0$$

or

$$1 - \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \tag{48}$$
$$\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq$$
$$\Big[- \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}$$
$$- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} +$$
$$\max_i \Big\{ E_{s_0}(MRS^i(c^i(s_1)) \, |s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} +$$
$$E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \Big\} \Big] f_k(\underline{s}_1^{nd})$$

where the term on the lhs is nonnegative because of (46) and the one on the rhs is also non-negative by construction. Analogously, substituting the expressions for the partial derivatives *into the FOC for* $dB = f_k(\underline{s}_1^{nd})dk < 0$ *yields:*

$$\Big[-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}+p\Big] f_k(\underline{s}_1^{nd})+$$
$$-1+\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}\geq 0$$

or

$$\begin{aligned}&\Big[-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}\\ &-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}+p\Big] f_k(\underline{s}_1^{nd})\\ \geq\ & 1-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}-\\ &\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}\end{aligned} \tag{49}$$

where the term on the lhs is nonnegative because of (45) and the one on the rhs is also nonnegative as it immediately follows from (46). Putting (48) and (49) together,

$$1-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}\geq$$
$$\Big[-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}+p\Big] f_k(\underline{s}_1^{nd})\geq$$
$$\Big[-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}-$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}+p\Big] f_k(\underline{s}_1^{nd})\geq$$
$$1-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}-$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$

Since

$$
\begin{aligned}
&-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
&-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
&-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

it must be that (47) holds, where recall that

$$
\begin{aligned}
p = \max_i \Bigg\{ & E_{s_0}(MRS^i(c^i(s_1)) \big| \, s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} + \\
& E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \Bigg\}
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} = \\
&\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
&\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \bigg| \, s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

and

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$
$$= \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$

Note that conditions (45), (46) and (47) can be alternatively stated as:

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1 \in S^{nd\prime}\Big\} \Pr\{s_1 \in S^{nd\prime}\} +$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg| s_1 \in S^{d\prime}) \Pr\{s_1 \in S^{d\prime}\} \geq p \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\} +$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}.$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1 \in S^{nd\prime}\Big\} \Pr\{s_1 \in S^{nd\prime}\} +$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg| s_1 \in S^{d\prime}) \Pr\{s_1 \in S^{d\prime}\} \geq 1 \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\} +$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}$$

and

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1 \in S^{nd}\Big\} \Pr\{s_1 \in S^{nd}\}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\} =$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg| s_1 \in S^{d}) \Pr\{s_1 \in S^{d}\}.$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$
$$+\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\} =$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$

This completes the proof of the claim. ■

We are now ready to complete the proof of the proposition. The equity price map in the presence of risky debt is giiven n by

$$q(k,B) = \max_i E_{s_0}\left\{ MRS^i(c^i(s_1))\,[f(k;s_1)-B]\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$

The debt price map is

$$p(k,B) = \max_i \Big\{ E_{s_0}(MRS^i(c^i(s_1))\big|\, s_1\in S^{nd})\Pr\{s_1\in S^{nd}\} +$$
$$+\, E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}\Big\}$$

The statement only refers to the interior case: $f(k;\underline{s}_1^{nd}) > B$. *Here, the partials of the price maps with respect to* B *are*

$$\frac{\partial q}{\partial B_+} = -\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$
$$\frac{\partial q}{\partial B_-} = -\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$

and

$$\frac{\partial p}{\partial B_+} = -\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$
$$\frac{\partial p}{\partial B_-} = -\max_{i\in I^d}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg|\, s_1\in S^d)\Pr\{s_1\in S^d\}$$

Analogously, the partials with respect to k *are*

$$\frac{\partial q}{\partial k_+} = \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$
$$\frac{\partial q}{\partial k_-} = \min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\big|\, s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$

and

$$\frac{\partial p}{\partial k_+} = \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}$$
$$\frac{\partial p}{\partial k_-} = \min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}$$

So, if $f(k;\underline{s}_1^{nd})>B$, the FOCs with respect to B are:

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+}B + p\right) =$$
$$-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$+\left(-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}+p\right)\le 0$$
$$\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-}B + p\right) =$$
$$-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$+\left(-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}+p\right)\ge 0$$

which implies

$$p = \max_i E_{s_0}\left(MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right)\Pr\{s_1\in S^{nd}\}+$$
$$E_{s_0}\left(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d\right)\Pr\{s_1\in S^d\}$$

and (43). On the other hand, the FOCs with respect to k give:

$$\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+}B\right) =$$
$$= -1 + \max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)B\Pr\{s_1\in S^d\}\le 0$$
$$\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-}B\right) =$$
$$= -1 + \min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)B\Pr\{s_1\in S^d\}\ge 0$$

which implies (44). This completes the proof of Proposition 2. ■

Equilibria with Short sales when intermediation costs are negligible

In Section 6 we established the existence of an equilibrium with intermediated short sales for all levels $\delta > 0$ of the intermediation cost (capturing the default rate on short positions). It is then of interest to investigate the properties of these equilibria as we let δ go to 0. Clearly the spread $\max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k,\phi,m,B;s)\right] - \min_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k,\phi,m,B;s)\right]$ must go to zero, since $q(k,\phi,m,B)$ is bounded above for all k,ϕ,m,B and all $\delta > 0$, total resources being finite. We conjecture therefore that the limit of the competitive equilibria with short sales as $\delta \to 0$ exists, as all variables lie in a compact set.

The previous observation also implies that the marginal valuation for all possible production and financial plans is equalized across all consumers, as in an environment where unlimited short sales are allowed and markets are complete (or a spanning property holds for all admissible production and financial plans of firms). In the limit as $\delta \to 0$ not only all possible markets, corresponding to all possible choices k,ϕ,m,B, are open, as in the case without short sales, but a òarger set of markets are open and active, to ensure the equalization of agents' marginal rates of substitution.

Short sales with endogenous default

We extend here the analysis of Section 6 by examining the case where the consumers' default rate, rather than being exogenous and state and type invariant, is optimally chosen by consumers, and may depend therefore on the state s as well as the type i of the consumer. We show in what follows the required changes in the model. The specification of the intermediation activity and the structure of markets is clearly more complicated, still the main results on unanimity and optimality remain valid.

Since consumers' loans are non-collateralized, we follow Dubey, Geanakoplos and Shubik (2005) in introducing a utility penalty ξ^i for a type i consumer per unit defaulted in any state s, for all i,s. It is convenient to assume here that preferences are additively separable over time, so that they take the following form:

$$u_0^i(c_0^i) + \mathbb{E}\left[u_1^i\left(c^i(s)\right) - \xi^i\delta_s^i\left[\lambda_-^i\left(f(k,\phi;s)-B\right)\right]\right] \qquad (50)$$

where δ_s^i is the default rate of consumer i in state s. Given this feature of consumers' preferences, the optimal default level in each state s for consumer i is obtained by maximizing (50) with respect to $(\delta_s^i)_s$ subject to the date 1 budget constraint (31), where δ is replaced by δ_s^i. It is immediate to see that the solution is a well defined map $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$ for all s and $\theta^i, \lambda_+^i, b^i, \lambda_-^i$, and for any given k, ϕ, B.

Thus the default rate in any state s on the loans granted to consumers via the sale of short positions depends not only on the type i of the consumer but also on his overall portfolio holdings. We consider then the case where both the consumer's type and his portfolio holdings are observable by his trading partners. The loan contract offered by intermediaries is so an exclusive contract and the price depends both on the consumer's type and portfolio, $q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}$ as well as, obviously, on the return structure of the underlying equity. Hence the budget constraint faced by consumers at date 0 is now

$$c_0^i \; = w_0^i \; + [-k \; + q \; + p \; B \;] \, \theta_0^i - q \; \theta^i \; - p \; b^i - q^+ \lambda_+^i + q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i} \lambda_-^i \tag{51}$$

An intermediary who is intermediating m units of the derivative by selling the short positions to consumers of type i, with portfolio $(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$, faces a default rate on its loans equal to $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$. As a consequence, the shortfall in its revenue at date 1 is:

$$\left[(f(k, \phi; s) - B) \;\; \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \right] m. \tag{52}$$

We consider still the case where only equity, an asset that is 'safe' as it is in positive net supply and backed by real claims, is used to hedge the consumers' default risk. To be able to fully meet the shortfall in (52) due to consumers' default, the intermediary must hold at least

$$\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) m$$

units of equity. The total date 0 revenue of the intermediary is then:

$$\max_m \left[q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i} - q \left(\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \right) \right] m \tag{53}$$

The intermediary's choice problem consists in the choice of the amount m to issue of each type $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$ of derivative so as to maximize its profits, that is its revenue at date 0. Note that the intermediation technology still exhibits constant returns to scale, hence a solution exists provided

$$q \geq \frac{q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)};$$

and is characterized by $m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) > 0$ only if $q = \frac{q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}}{\max_s \delta_s^i(\theta^i,\lambda_+^i,b^i,\lambda_-^i)}$.

The main difference with respect to the reduced form model is then the fact that the market for derivative claims is differentiated according to consumers' types and portfolio choices. This has the following implications for the consumers' optimization problem and the market clearing conditions.

Consumer i chooses his portfolio $\theta^i, \lambda_+^i, b^i, \lambda_-^i$ so as to maximize

$$u_0^i(c_0^i) + \mathbb{E}\left\{ \begin{array}{c} u_1^i\left[w^i(s) + b^i + (f(k,\phi;s) - B)(\theta^i + \lambda_+^i - \lambda_-^i(1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i))\right] \\ -\xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \end{array} \right\}$$

subject to the budget constraint (51), given the asset prices q, q^+, p and $q_{i,\cdot}^-$ and the default map $\delta_s^i(\cdot)$ obtained as above. Let $\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i$ denote the consumer's optimal choice in equilibrium. The asset market clearing conditions are then

$$\sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i)\left[\max_s \delta_s^i(\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i)\right] + \sum_{i\in I} \bar{\theta}^i = 1$$

for equity, and

$$\begin{aligned} \bar{\lambda}_-^i &= m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \text{ for each } i \\ 0 &= m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) \text{ for each } i,\ (\theta^i, \lambda_+^i, b^i, \lambda_-^i) \neq (\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \\ \sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) &= \sum_i \bar{\lambda}_+^i \end{aligned}$$

for the derivative security.

The consistency condition M) on the firms' equity conjectures must also be properly modified to reflect the different specification of the value of intermediation in the present context:

$$M')\ \ q(k,\phi,B) = \max \left\{ \begin{array}{c} \max_i \mathbb{E}\left[MRS^i(c^i(s))\,(f(k,\phi;s) - B)\right], \\ \max_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i} \frac{\max_i \mathbb{E}\left[MRS^i(c^i(s))(f(k,\phi;s)-B)\right] - q^-(i,\theta^i,\lambda_+^i,b^i,\lambda_-^i;k,\phi,\bar{U}^i)}{\max_s \delta_s^i(\theta^i,\lambda_+^i,b^i,\lambda_-^i;k,\phi)} \end{array} \right\}, \forall k, \phi, B$$

where $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ is constructed as follows. For any k, ϕ, B and $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$, set $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ as the value of q^- that satisfies the following equation:

$$\begin{aligned} \bar{U}^i &= u_0^i(w_0^i + \left[-\bar{k} + \bar{q} + \bar{p}\,\bar{B}\right]\theta_0^i - \bar{q}\,\theta^i - \bar{p}\,b^i - \bar{q}^+\lambda_+^i + q^-\lambda_-^i) + \\ & \mathbb{E}\left[\begin{array}{c} u_1^i\left[w^i(s) + b^i + (f(\bar{k},\bar{\phi};s) - \bar{B})(\theta^i + \lambda_+^i) - \lambda_-^i\left[f(k,\phi;s) - B\right]\left(1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)\right)\right] \\ -\xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)\left[\lambda_-^i\left(f(k,\phi;s) - B\right)\right] \end{array} \right. \\ &= \bar{U}^i \end{aligned}$$

where $\bar{U}^i$ denotes the utility level of type i consumers at the equilibrium choices $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{b}^i, \bar{\lambda}^i_-$ and the map $\delta^i_s(\theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B)$ is similarly obtained by maximizing the expected utility term on the right hand side of the above expression with respect to δ^i_s. That is, $q^-(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B, \bar{U}^i)$ identifies the maximal willingness to pay in equilibrium of consumer i for a short position equal to λ^i_- in the firm with plan k, ϕ, B when the rest of his portfolio is given by $\theta^i, \lambda^i_+, b^i$.[59] At these prices consumers are indifferent between choosing the equilibrium portfolio $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{b}^i, \bar{\lambda}^i_-$ and any other portfolio with a short position λ^i_- in the equity of a firm with plan k, ϕ, B.

An important difference with respect to the previous analysis is the fact that here the price of short positions is no longer defined at the margin. This is due to the exclusive nature of loan contracts corresponding to short positions. Also, at the same prices intermediaries are indifferent between issuing the derivatives traded in equilibrium and any other derivative on equity of firms with plan k, ϕ, B such that $q = \frac{\max_i \mathbb{E}\left[MRS^i(c^i(s))(f(k,\phi;s)-B)\right] - q^-(i,\theta^i,\lambda^i_+,b^i,\lambda^i_-;k,\phi,B,\bar{U}^i)}{\max_s \delta^i_s(\theta^i,\lambda^i_+,b^i,\lambda^i_-;k,\phi,B)}$.

The unanimity and constrained optimality properties still hold in this environment. The argument again is very similar and relies on the the fact that, given the above specification of the intermediation technology and the price conjectures, the model is equivalent to a setup where the markets for all types of equity and all types of corresponding derivatives are available for trade. The notion of completeness here also requires the exclusivity of the loan contracts associated to short positions, so that the market for all types of derivatives can also be differentiated according to the type of a consumer and the level of his trades.

[59] In the specification of $q^-(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B, \bar{U}^i)$ we have implicitly assumed that all the long positions of a consumer are in the assets corresponding to the firms' equilibrium choices. This is with no loss of generality and to avoid excessive notational complexities.