Competing Mechanisms in Markets for Lemons*

Sarah Auster‡ Piero Gottardi§

March 2018

Abstract

We study competitive search equilibria in a decentralized market with adverse selection, where uninformed buyers post general trading mechanisms and informed sellers select one of them. We show that this has different, significant implications with respect to the traditional approach, based on bilateral contracting between the parties: in equilibrium all buyers post the same mechanism and low quality sellers receive priority in any meeting with a buyer. Also, buyers make strictly higher profits with low than with high type sellers. When adverse selection is severe, the equilibrium features rationing and is constrained inefficient. Compared to the equilibrium with bilateral contracting, the equilibrium with general mechanisms yields a higher surplus for most, but not all, parameter specifications. Hence in some situations restricting the set of available mechanisms is welfare improving.

1 Introduction

Spurred by recent events in financial markets, such as market freezes and the rapid growth of decentralized forms of trading, there has been a renewed interest in the study of markets with adverse selection and with richer trading processes. The hiring process in the labor

*We would like to thank Willie Fuchs, Andrea Galeotti, Veronica Guerrieri, Teddy Kim, Philipp Kircher, Nenad Kos, Ben Lester, Andrea Mattozzi, Sergei Severinov, Pierre-Olivier Weill, Randy Wright and Ronald Wolthoff for very helpful comments. We gratefully acknowledge financial support from grant MIUR - PRIN Bando 2015 Prot. 20157NH5TP.

‡Department of Decision Sciences and IGIER, Bocconi University, email: sarah.auster@unibocconi.it

§Department of Economics, European University Institute, email: piero.gottardi@eui.eu

market has also traditionally been viewed as occurring in a decentralized fashion and, in some cases, characterized by the presence of adverse selection. The traditional approach to markets with adverse selection relies on bilateral contracting between the parties, where the trading protocol involves one principal and one agent. In many situations, however, the process leading to a transaction features multiple agents interacting with the same principal. In decentralized financial markets buyers may be approached by several sellers at the same time: for instance, a private equity firm negotiates the possibility of an investment with several entrepreneurs. In the labor market the analysis of the empirical evidence on these issues has been the subject of various papers. In particular, the recent work by Davis (2017) provides new support in favor not only of the presence of multiple applications but also of the fact that they are processed at the same time, not in sequence (see also De Los Santos et al. (2012) for other evidence in support of this from consumers' markets).

This paper studies the properties of competitive markets with adverse selection when the trading protocol allows for the possibility that several agents interact with the same principal. We argue this is important in the light of the previous considerations. Moreover, even though multiple interactions had been earlier considered in markets with public information or independent private values, we will show that, in the presence of adverse selection, allowing for multiple interactions has significant and novel implications for the properties of market outcomes, in particular their welfare features.

In markets with adverse selection, as well known, uninformed principals have an incentive to use the terms of trade they post to separate the different types of agents. Separation can occur in two ways: within the offer made by a single principal (when this entails a menu of contracts, or a trading mechanism), or by having different types self selecting into different offers posted. A common finding in the literature following the traditional approach with bilateral contracting recalled below is that the latter always obtains in equilibrium: agents are separated in their choice of contract and principals' profits are then equalized across the different contracts posted. Hence the equilibrium exhibits no cross subsidization, which has important welfare implications with adverse selection. We will show that this feature is overturned when we allow for multiple interactions, hence the equilibria may now feature cross subsidies across types but also must satisfy other properties, which make the welfare comparison in the two set-ups far from trivial and quite rich.

More specifically, in the model we consider uninformed principals post trading mechanisms, informed agents select a mechanism and a principal posting it. We therefore follow a directed search approach, where each principal may be selected by several or no agents.

The distinctive feature is that we allow principals to exploit this fact by posting general direct mechanisms that specify trading probabilities and transfers for agents, contingent not only on their own reported type but also on the reports of other agents meeting the same principal. By doing so principals make agents compete among themselves and induce them to reveal information about their types. As an illustration we can think of a trading mechanism as a form of auction. The approach we follow appears natural in situations like decentralized procurement markets, where procurers meet several firms, privately informed about the quality of their services. In this context, the terms of trade between a procurer and a contractor may not only depend on the contractor's type but also on the number and composition of the contractor's competitors. More generally, as already mentioned above, this approach is also relevant for other decentralized markets, where trading protocols are less explicit but still involve multiple agents, like in the case of financial markets, but also in the labor market, where firms meet a variety of workers, privately informed about their productivity. In addition to the empirical support we discussed regarding the importance of considering trading mechanisms involving several agents, we should mention some theoretical considerations too. The existing microfoundations of the competitive search equilibrium concept (see, e.g. Peters (1997)) consider finite games where principals propose terms of trade and agents select one of the principals. Hence each principal may meet several agents and it is natural to consider the case where the terms of trade he posts exploit this fact.

In an environment as in Akerlof (1970), where sellers are privately informed about the quality of their good, which can be high or low, we find that there always exists an equilibrium in which all buyers post and hence all sellers select the same mechanism. In the situation we consider buyers make strictly higher profits with low than with high type sellers. Cream-skimming deviations, attracting only the high types, turn then to be not profitable, which makes it possible to sustain pooling of high and low type sellers on the same mechanism. The equilibrium mechanism specifies that a low quality object is traded whenever such object is present in a meeting between a buyer and one or more sellers, meaning that low type sellers receive priority in every meeting. Remarkably, this property holds no matter how large the gains from trade for the high quality good are relative to those for the low quality good. We show that the equilibrium mechanism can be implemented via a sequence of second price auctions with increasing reserve prices.

While the properties of the equilibrium allocation described above hold regardless of the size of the gains from trade for low and high quality goods, the latter matters for other features of the equilibrium. We find that in situations where buyers care a lot for quality and competition among them for high quality sellers is sufficiently intense - a situation we refer to as adverse selection being severe - buyers make zero profits with high quality sellers and meetings where only those sellers are present do not always end with trade. The intuition is that giving priority to low quality sellers might not be sufficient to satisfy their incentive compatibility constraint, so the trading probability of sellers with a high quality good needs to be reduced further by rationing them in meetings where they are alone. Also, in such case additional equilibria exist where sellers partially separate themselves through their choice of mechanism: buyers post different mechanisms, attracting different ratios of high types to buyers. A sufficiently low trading probability for high quality sellers can therefore be achieved either via rationing, within the mechanism, or through an asymmetric assignment of high type sellers across the mechanisms posted by buyers, or a combination of the two. These are new properties due to adverse selection.

The extent to which adverse selection is or is not severe also matters for the welfare properties of the equilibrium. We show that, whenever adverse selection is not severe and the gains from trade are higher for the low than the high quality object, the equilibrium allocation maximizes total surplus. The result follows directly from the facts that (i) the pooling of sellers on the choice of the same mechanism maximizes the number of meetings and that (ii) the equilibrium mechanism gives priority to the good with the larger gains from trade. On the other hand, with severe adverse selection total surplus is no longer maximal in equilibrium and, provided the share of high type sellers is large enough, the equilibrium allocation can be Pareto improved, even subject to the constraints imposed by incentives and the meeting technology.

Finally, it is of interest to compare in our environment the welfare properties of the equilibria we found to those of the equilibria that obtain when contracting can only be bilateral. To this end in the last part of the paper we consider the case where the mechanisms available to buyers are restricted to the class of simple menus from which one seller, randomly selected among those meeting a buyer, can choose. As one should expect in the light of the results of earlier work (see for example Guerrieri et al., 2010), we find that in equilibrium buyers post two distinct prices, the higher one selected by high type sellers and the lower one by low types. We then see that total surplus at the equilibrium with general mechanisms is strictly higher than at the one with bilateral contracts for many, but not all, parameter specifications. Hence there are situations where introducing restrictions on the set of available mechanisms proves welfare improving. Even more surprisingly, this does not arise in the case of severe adverse selection where the equilibrium with general mechanisms features rationing. It occurs instead when gains from trade for the high quality good are larger than for the low quality good but competition for high type sellers is sufficiently weak so that every meeting still leads to trade.

Related Literature: Since the work of Akerlof (1970) and Rothschild and Stiglitz (1976), the properties of market outcomes in the presence of adverse selection have been the subject of intense study, both in Walrasian models as well as in models where agents act strategically. In the latter agents compete among themselves over contracts which determine transfers of goods and prices, whereas in the former available contracts specify transfers of goods while prices are taken as given and set so as to clear the markets. Initiated by Gale (1992), Inderst and Mueller (1999) and, more recently, Guerrieri et al. (2010), the use of competitive, directed search models to study markets with adverse selection has generated several interesting insights.[1] This framework allows for a richer specification of the terms of the contracts available for trade, which - in contrast to Walrasian models - also include the price to be paid. Both in competitive search models and in Walrasian ones (on the latter, see Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006)), it was shown that when contracting is bilateral an equilibrium always exists and is separating, with agents of different types trading different contracts.[2]

Our paper is also closely related to the work on competing mechanisms in independent private value environments. Peters (1997) and Peters and Severinov (1997) consider a framework where buyers are not constrained in their ability to meet and contract with multiple sellers, as we do, and show that an equilibrium exists where all buyers post the same mechanism given by a second-price auction with a reserve price equal to their valuation.[3] The finding that a single mechanism is posted in equilibrium is analogous to what we obtain in the presence of common value uncertainty, though the posted mechanism is different from a second-price auction in several important aspects. Furthermore, the argument to establish

[1]For earlier work on directed search in markets without informational asymmetries see, for instance, Moen (1997), Burdett et al. (2001).

[2]There are also papers studying adverse selection in economies where meetings and trades take place over a sequence of periods, both in markets with random search, where offers of contracts are made only after meetings occur, and in markets à la Akerlof (1970), where all trades occur at a single price (see for example Blouin and Serrano (2001), Janssen and Roy (2002), Camargo and Lester (2014), Fuchs and Skrzypacz (2013) and Moreno and Wooders (2016)). It is interesting to notice that the equilibrium outcome is similar in these dynamic models: separation in that case obtains with sellers of different types trading at different prices, at different points in time.

[3]Note that in this literature labels of buyers and sellers are typically reversed.

the property that all buyers post the same mechanism is rather different in our case. A useful property in independent private value settings is that equilibria are efficient. One can therefore solve the planner's problem, find a way to decentralize the solution, and show that no profitable deviations exist.[4] In our environment with adverse selection, however, this approach cannot be followed since we cannot rely on the fact that the equilibrium is efficient. Nevertheless we show that, once the general space of mechanisms is projected to a much simpler reduced space, the characterization of the equilibrium is still tractable and is obtained by directly solving the buyers' optimization problem. Another important difference with respect to the independent private value case is the fact that in our environment rationing and partial separation of sellers across different mechanisms can occur in equilibrium, two features that never arise when buyers' valuations do not depend on sellers' types.

Eeckhout and Kircher (2010) study the role of the properties of the meeting technology for the allocation that obtains at a competitive search equilibrium, again in a setting with independent private values.[5] In particular they show that, when meetings can only be bilateral, in equilibrium buyers post different prices attracting different types of sellers, whereas such equilibria do not exist when a seller's probability of meeting a buyer is not affected by the presence of other sellers. These results are related to our discussion of equilibria when mechanisms are restricted to simple menus, as these constitute the available mechanisms when buyers meet at most one seller. Also the literature on directed search with public information has considered different assumptions on the space of available mechanism and obtained similar results: if agents can be discriminated by the mechanism through a priority rule, different types will be pooled on the same mechanism (see Shi (2002) and Shimer (2005)), whereas if the set of available mechanism is restricted to the set of posted prices, equilibria are separating (see Shi (2001)). In contrast to these papers, however, our focus lies on the welfare implications of such restriction, for which adverse selection plays a critical role, as we show in Section 4.1.

The paper is organized as follows. The next section presents the economy and the set of available trading mechanisms and defines the notion of competitive search equilibrium that is considered. Section 3 establishes the main result, showing the existence and the uniqueness of the equilibrium allocation and characterizing its properties, presenting the key steps of the proof. Section 4 provides the welfare analysis of the equilibria we found and compares their properties to those of the equilibrium which obtains when mechanisms are restricted to

[4]See, for instance, Cai et al. 2017a and Lester et al. 2017b. Albrecht et al. (2014) consider an environment with free entry of the uninformed party and show that the equilibrium remains efficient.

[5]Their findings were further developed by Lester et al. 2015 and Cai et al. 2017a.

bilateral contracts, showing that welfare is typically, though not always, higher in the first one. Proofs are collected in the Appendix.

2 Environment

There is a measure b of uninformed buyers and a measure s of informed sellers. Each seller possesses one unit of an indivisible good with uncertain quality and each buyer wants to buy at most one unit. The good's quality is identically and independently distributed across sellers. Quality can be either high or low and μ denotes the fraction of sellers that possess a high quality good. Let $\overline{\lambda}^p = \mu \frac{s}{b}$ denote the ratio of high type sellers to buyers and $\underline{\lambda}^p = (1-\mu)\frac{s}{b}$ the ratio of low type sellers to buyers.

The valuation of buyers and sellers for the low quality good is denoted, respectively, by $\underline{v}$ and $\underline{c}$, that of the high quality good by $\overline{v}$ and $\overline{c}$. We assume that both the buyers and the sellers value the high quality good more than the low quality good, i.e. $\overline{v} \geq \underline{v}$, $\overline{c} > \underline{c}$. For sellers this preference is assumed to be strict, while we allow buyers to have the same valuation for both types of good, i.e. $\overline{v} = \underline{v}$. When $\overline{v}$ is strictly greater than $\underline{v}$, the buyer's valuation depends on the seller's valuation of the object, a situation we refer to as the common value case. This is no longer true when $\overline{v} = \underline{v}$, which we refer to as the private value case. We further assume that there are always positive gains from trade, meaning that for both types of the good the buyer's valuation strictly exceeds the seller's valuation, i.e. $\overline{v} > \overline{c}$, $\underline{v} > \underline{c}$.

Meeting Technology: The trading process operates as follows. Buyers simultaneously post mechanisms that specify how trade takes place with the sellers they meet. Sellers observe the posted mechanisms and select one of the mechanisms they like best as well as one of the buyers posting it. We refer to the collection of buyers posting the same mechanism and the collection of sellers selecting that mechanism as constituting a submarket. We have in mind markets that are decentralized and anonymous, where interactions are relatively infrequent, as in the situations mentioned in the Introduction. In line with the first feature, the matching between buyers and sellers in any submarket does not occur in a centralized way but is determined by the individual choices of sellers, each of them selecting a buyer. Anonymity is captured by the assumption that the mechanisms posted do not condition on the identity of sellers and that sellers do not condition their choice on the identity of buyers, only on the mechanism they post, and thus pick at random one of the buyers posting the selected mechanism (see for example Shimer, 2005). Also, since each seller can only select one buyer, contracting is exclusive.

More specifically, we assume that, in any submarket, a seller meets one, randomly selected, of the buyers that are present. Buyers have no capacity constraint in their ability to meet sellers, that is, they can meet all arriving sellers, no matter how many they are. As a result, matching occurs according to the urn-ball technology where the number of sellers that meet a particular buyer follows a Poisson distribution with a mean equal to the seller-buyer ratio in the submarket, also referred to as the queue length of that submarket.[6] According to this meeting technology, sellers are sure to meet a buyer, while buyers may end up with many sellers or with no seller at all. Moreover, a buyer's probability of meeting a seller of a given type is fully determined by the ratio between sellers of that type and buyers in the submarket, while it does not depend on the presence of other types of sellers.[7] The latter property and the fact that buyers can meet multiple sellers are essential for the following analysis, while most other features of the meeting technology are not. Also, the fact that in the market there is one-sided heterogeneity, allowing for one-to-many meetings only matters when it is the buyer that can meet several heterogenous sellers.[8]

Under urn-ball matching, a buyer's probability of meeting k sellers in a market with queue length λ is given by

$$P_k(\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

Since the presence of high type sellers does not affect the meeting chances of low type sellers and vice versa, the probability for a buyer to meet L low type sellers and H high type sellers in a market where the queue length of low and high type sellers is, respectively, $\underline{\lambda}$ and $\overline{\lambda}$ is given by

$$P_L(\underline{\lambda})P_H(\overline{\lambda}) = \frac{(\underline{\lambda})^L}{L!}e^{-\underline{\lambda}}\frac{(\overline{\lambda})^H}{H!}e^{-\overline{\lambda}}.$$

Similarly, $P_L(\underline{\lambda})P_H(\overline{\lambda})$ corresponds to the probability for a seller to be in a meeting with other L low type sellers and H high type sellers.

Mechanisms and payoffs: We restrict attention to direct mechanisms that do not

[6]The urn-ball matching technology is one of the most commonly used specifications in the directed search literature, e.g. Peters and Severinov (1997), Albrecht et al. (2006), Kim and Kircher (2015).

[7]The class of meeting technologies that have this property, called 'invariance' (Lester et al., 2015), includes the urn-ball matching technology as a special case.

[8]See the concluding remarks for further discussions of possible generalizations of the meeting technology.

condition on mechanisms posted by other buyers.[9] A mechanism m is defined by the maps

$$\left(\underline{X}_m, \overline{X}_m, \underline{T}_m, \overline{T}_m\right) : \mathbb{N}^2 \to [0,1]^2 \times \mathbb{R}^2,$$

where the first argument is the number of low messages and the second argument is the number of high messages received by a buyer from the sellers he meets. The maps $\underline{X}_m(L,H), \overline{X}_m(L,H)$ and $\underline{T}_m(L,H), \overline{T}_m(L,H)$ describe the trading probabilities and transfers to a seller (unconditional on whether trade occurs or not) specified by mechanism m for sellers reporting, respectively, to be of low and high type. That is, $\underline{X}_m(L,H)$ is the trading probability for a seller who reports to be of low type when $L-1$ other sellers report to be of low type and H other sellers report to be of high type. We say a mechanisms m is feasible if

$$\underline{X}_m(L,H)L + \overline{X}_m(L,H)H \leq 1, \forall (L,H) \in \mathbb{N}^2. \tag{1}$$

Thus, in each meeting the probability that a good is exchanged cannot exceed one. Let M denote the measurable set of feasible mechanisms.

We assume that, when matched with a buyer, a seller does not observe how many other sellers are actually matched with the same buyer nor their types.[10] Let $\overline{\lambda}$ denote the expected number of high reports and $\underline{\lambda}$ the expected number of low reports, which under truthful reporting simply correspond to the respective seller-buyer ratios, for mechanism m. The expected trading probabilities for a seller when reporting to be of low and of high type,

[9] In principle, indirect mechanisms could allow buyers to elicit information about the mechanisms other buyers post. However, given the large market environment we consider, in the equilibria we characterize in Theorem 3.1 below each buyer knows the realized distribution of mechanisms posted by other buyers and this distribution is not affected by deviations of individual buyers. It then follows that, as noted by Eeckhout and Kircher (2010) in an analogous set-up, the buyer does not profit from conditioning on those mechanisms. Hence, the equilibria we characterize are robust to removing the restriction to mechanisms conditioning only on the sellers' reported types, though additional equilibria may exist in that case.

[10] The assumption that a seller cannot observe the realized number of competitors in a meeting facilitates notation considerably but is not essential for any of our results. To see this notice that, since traders are assumed to be risk neutral, sellers' payoffs are linear in trading probabilities and transfers. Therefore, if incentive compatibility constraints (2) and (3) below are satisfied, the values of the expected trading probabilities and transfers can always be decomposed in terms of trading probabilities and transfers in each possible meeting in such a way that incentive compatibility is satisfied also when sellers observe the number of competitors in a meeting.

respectively, are then given by

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\underline{X}(L+1, H),$$
$$\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\overline{X}(L, H+1).$$

Similarly, we can determine expected transfers $\underline{t}_m(\underline{\lambda}, \overline{\lambda})$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda})$. The expected payoff for low and high type sellers when choosing mechanism m and revealing their type truthfully, net of the utility of their endowment point,[11] is then given by

$$\underline{u}(m|\underline{\lambda}, \overline{\lambda}) = \underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c},$$
$$\overline{u}(m|\underline{\lambda}, \overline{\lambda}) = \overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c}.$$

Truthful reporting is optimal if the following two inequalities hold

$$\overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c} \leq \underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c}, \tag{2}$$
$$\underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c} \leq \overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c}. \tag{3}$$

Note that, since incentive compatibility is defined in terms of expected trading probabilities and transfers, whether a given mechanism m is incentive compatible or not depends on the values of $\underline{\lambda}$ and $\overline{\lambda}$. Let $\mathcal{M}^{IC}$ denote the set of tuples $(m, \underline{\lambda}, \overline{\lambda})$ such that $m \in M$ and incentive compatibility with respect to $\underline{\lambda}, \overline{\lambda}$ is satisfied.

The expected payoff for a buyer posting mechanism m, when sellers report truthfully and the expected number of high and low type sellers, respectively, is $\overline{\lambda}$ and $\underline{\lambda}$, is then

$$\pi(m|\underline{\lambda}, \overline{\lambda}) = \overline{\lambda}[\overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{v} - \overline{t}_m(\underline{\lambda}, \overline{\lambda})] + \underline{\lambda}[\underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{v} - \underline{t}_m(\underline{\lambda}, \overline{\lambda})].$$

Equilibrium: An allocation in this setting is defined by a probability measure β over M with support M^β, where $\beta(M')b$ describes the measure of buyers that post mechanism $M' \subseteq M$, and two maps $\underline{\lambda}, \overline{\lambda} : M^\beta \to \mathbb{R}^+$ specifying, respectively, the ratio of low and high type sellers selecting mechanism m relative to the buyers posting that mechanism. We say

[11]For example, the endowment point of a low type seller is $\underline{c}$. His utility gain when participating in mechanism m is then given by $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) + (1 - \underline{x}_m(\underline{\lambda}, \overline{\lambda}))\underline{c} - \underline{c} = \underline{u}(m|\underline{\lambda}, \overline{\lambda})$.

an allocation is *feasible* if [12]

$$\int_{M^\beta} \underline{\lambda}(m) d\beta(m) = \underline{\lambda}^p, \quad \int_{M^\beta} \overline{\lambda}(m) d\beta(m) = \overline{\lambda}^p. \tag{4}$$

We call an allocation *incentive compatible* if $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$ for all $m \in M^\beta$. We can show that we can restrict our attention to incentive compatible allocations w.l.o.g.:[13] for any non-incentive compatible mechanism there always exists an incentive compatible mechanism that yields the same payoff for buyers and sellers as the original mechanism.

An equilibrium in the environment considered is given by a feasible and incentive compatible allocation such that the values of β and $\underline{\lambda}, \overline{\lambda}$ are consistent with buyers' and sellers' optimal choices. For mechanisms not posted in equilibrium, we extend the maps $\underline{\lambda}, \overline{\lambda}$ to the domain $M \backslash M^\beta$ in order to describe the beliefs of buyers over the queue lengths of low and high type sellers, respectively, that such mechanisms attract. We require that these beliefs are pinned down by a similar consistency condition with sellers' optimal choices out of equilibrium. More specifically, a buyer believes that a deviating mechanism attracts some low type sellers only if low type sellers are indifferent between this mechanism and the one they choose in equilibrium, and similarly for high type sellers.[14]

$$\begin{aligned} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m'\beta} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0, \quad (5) \\ \overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m'\beta} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0. \quad (6) \end{aligned}$$

Hence we can say beliefs are correct also out of equilibrium, were a deviation to occur. This specification of the beliefs for mechanisms that are not posted in equilibrium is standard in the literature of directed search, both with and without common value uncertainty (see Guerrieri et al. (2010) and Eeckhout and Kircher (2010) among others).

Formally, we impose the following conditions on out of equilibrium beliefs, $\underline{\lambda}(m), \overline{\lambda}(m), m \notin M^\beta$:

[12]Let $\bar{\sigma}(M')$ denote the probability with which high type sellers search for mechanisms in $M' \subseteq M$ and assume that $\bar{\sigma}$ is absolutely continuous in β (sellers can only search for mechanisms that are posted by some buyers). Condition 4 below says that the ratio $\bar{\lambda}$ is obtained as the product of $\bar{\lambda}^p$ and the Radon-Nikodym derivative of the $\bar{\sigma}$ with respect to β. Similar for the ratio $\underline{\lambda}$.

[13]For a formal proof see the Online Appendix, available at https://sites.google.com/site/austersarah/

[14]Conditions (5),(6) below are perfectly analogous to the sellers' optimality conditions appearing in the equilibrium Definition 1. Hence (5),(6) indeed require the seller-buyer ratios are consistent with sellers' optimal choices, also for out of equilibrium mechanisms, as if all mechanisms were effectively available to sellers. This is analogous to existing refinements in other competitive models with adverse selection such as Gale (1992) and Dubey and Geanakoplos (2002).

i) if (5,6) admit a unique solution, then $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are given by that solution;

ii) if (5,6) admit no solution, we set $\underline{\lambda}(m)$ and/or $\overline{\lambda}(m)$ equal to $+\infty$ and $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) = c$ for some $c \leq 0$;[15]

iii) if (5,6) admit multiple solutions, then $\underline{\lambda}(m), \overline{\lambda}(m)$ are given by the solution for which the buyer's payoff $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m))$ is the highest.

Condition i) says that whenever there is a unique pair $\underline{\lambda}(m), \overline{\lambda}(m)$ satisfying conditions (5) and (6) for an out of equilibrium mechanism $m \notin M^{\beta}$, buyers' beliefs regarding the queue lengths for such deviating mechanism are pinned down by these conditions. This includes the solution where both (5) and (6) are satisfied with strict inequality and $\underline{\lambda}(m) = \overline{\lambda}(m) = 0$, in which case the buyer believes that m does not attract any seller since both types of sellers are worse off by choosing mechanism m, whatever the queue length. Since we allow for arbitrary direct mechanisms which may feature participation transfers paid to a seller independently of whether or not he trades and buyers have no constraints in their ability to meet sellers, there are mechanisms for which a solution to (5,6) does not exist: that is, for any finite pair $\underline{\lambda}, \overline{\lambda}$, at least one type of seller strictly prefers the deviating mechanism over any mechanism in the support of M^{β}.[16] Condition ii) specifies that in such case the queue lengths $\underline{\lambda}(m), \overline{\lambda}(m)$ are set equal to infinity, while a buyer's associated payoff is non-positive. Finally, if a solution to (5,6) exists and is not unique we follow McAfee (1993), Eeckhout and Kircher (2010) and others and assume, in condition iii), that buyers are 'optimistic', so that the pair $\underline{\lambda}(m), \overline{\lambda}(m)$ is given by their preferred solution.[17]

We are now ready to define a competitive search equilibrium.

Definition 1. *A competitive search equilibrium is a feasible and incentive compatible allocation given by a probability measure β with support M^{β} and two maps $\underline{\lambda}, \overline{\lambda} : M \to \mathbb{R}^{+} \cup +\infty$ such that the following conditions hold:*

[15] More precisely, if $\underline{u}(m|\underline{\lambda}, \overline{\lambda}) > \max_{m'^{\beta}} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ and $\overline{u}(m|\underline{\lambda}, \overline{\lambda}) > \max_{m'^{\beta}} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ for all $\left(\underline{\lambda}, \overline{\lambda}\right) \in \mathbb{R}^2_+$, then $(\underline{\lambda}(m), \overline{\lambda}(m)) = (+\infty, +\infty)$. If only one of the two inequalities is violated, say the first one, then $\overline{\lambda}(m) = +\infty$, while $\underline{\lambda}(m)$ is determined by (6).

[16] If the participation transfer is large enough, all sellers strictly prefer the deviating mechanism regardless of how many other sellers are expected to be present in a meeting. Such a mechanism would attract infinitely many sellers, to each of whom a strictly positive transfer is made. Since the possible gains from trade of a buyer are finite, it follows that his payoff associated with such mechanism cannot be positive.

[17] For mechanisms such as posted prices or standard auctions the solution is indeed unique.

- *buyers' optimality: for all $m \in M$ such that $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$,*

$$\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) \leq \max_{m'^{\beta}} \pi(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } m \in M^{\beta};$$

- *sellers' optimality: for all $m \in M^{\beta}$*

$$\begin{aligned} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m'^{\beta}} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0, \\ \overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m'^{\beta}} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0; \end{aligned}$$

- *beliefs: for all $m \notin M^{\beta}$, $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are determined by conditions i)-iii).*

3 Competitive Search Equilibrium

We state now our main result, which characterizes the competitive search equilibria in the environment described in the previous section.

Theorem 3.1. *There exists a competitive search equilibrium with the following properties:*

i) All buyers post the same mechanism.

ii) Whenever a low type seller is present in a match with a buyer, a low quality good is traded.

iii) The equilibrium is unique in terms of expected payoffs.

Theorem 3.1 states that there always exists an equilibrium in which sellers are pooled at the stage of selecting a mechanism and screened at the mechanism stage. That is, all buyers post identical mechanisms so that everybody trades in a single submarket and this mechanism specifies different trading probabilities for different types of sellers. In particular, the equilibrium mechanism always gives priority to low type sellers, meaning that a low quality good is traded whenever a low type seller is present in a meeting with a buyer. This not only implies that a low type seller's probability of trade strictly exceeds a high type seller's probability of trade,[18] but also that the equilibrium allocation maximizes trade

[18]The probability of trade for a low type seller is strictly larger than his probability of meeting no other low type seller, while the probability of trade for a high type seller is strictly smaller than his probability of meeting no low type seller.

of low quality goods in the economy. It is important to point out that this property of the equilibrium holds whatever is the size of the relative gains from trade or the fraction of high type sellers in the population. That is, even when the gains from trade for the low quality good are arbitrarily small and those for the high quality good are arbitrarily large, in equilibrium high type sellers only trade when they are in meetings where there are no low type sellers. Theorem 3.1 also states that the equilibrium is unique in terms of payoffs. In particular, although we will show there may also be equilibria where more than one mechanisms is posted, all those equilibria yield the same expected levels of trade and transfers. The remainder of this section is devoted to proving the above result.

3.1 Buyers' Auxiliary Optimization Problem

A distinguishing feature of the environment considered, with asymmetric information and common value uncertainty, as noticed in the introduction, is that the equilibrium cannot be found by decentralizing the constrained planner's solution. To establish the existence and the stated properties of competitive search equilibria we must therefore study the solution of the buyers' optimization problem. Given that the space of mechanisms available to buyers is quite large, we first show that we can conveniently restrict our attention to the space of expected trading probabilities and transfers associated to mechanisms in M. To this end we provide a characterization of the set of expected trading probabilities and transfers associated to the feasible and incentive compatible mechanisms in our environment.[19] This makes the buyer's problem tractable and the study of its solution allows us to derive some properties of the equilibrium outcome and then to show the existence of an equilibrium.

More precisely, the following result provides conditions on the values of expected trading probabilities and transfers that any feasible and incentive compatible mechanism satisfies and, viceversa, that are generated by some feasible and incentive compatible mechanism.

Proposition 3.2. *For any* $(\underline{x}, \overline{x}, \underline{t}, \overline{t}) \in [0,1]^2 \times \mathbb{R}^2$ *and* $\overline{\lambda}, \underline{\lambda} \in [0, \infty)$*, there exists a feasible and incentive compatible mechanism* m *such that*

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \quad \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}, \quad \underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \quad \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t},$$

[19] Expected trading probabilities and transfers have been used in the earlier literature on directed search to determine equilibrium payoffs (see, for instance, Shi (2001 and 2002) for the case of public information and Peters (1997) for the case of independent private values), but the characterization we provide of this set is novel.

if and only if

$$\bar{t} - \overline{x}\,\underline{c} \;\leq\; \underline{t} - \underline{x}\,\underline{c}, \tag{7}$$

$$\underline{t} - \underline{x}\,\overline{c} \;\leq\; \bar{t} - \overline{x}\,\overline{c}, \tag{8}$$

$$\overline{\lambda}\,\overline{x} \;\leq\; 1 - e^{-\overline{\lambda}}, \tag{9}$$

$$\underline{\lambda}\,\underline{x} \;\leq\; 1 - e^{-\underline{\lambda}}, \tag{10}$$

$$\overline{\lambda}\,\overline{x} + \underline{\lambda}\,\underline{x} \;\leq\; 1 - e^{-\overline{\lambda}-\underline{\lambda}}. \tag{11}$$

Proof See Appendix A.1.

Conditions (7) and (8) are analogous to the sellers' incentive compatibility constraints (2) and (3). It is immediate to see that these two conditions imply $\underline{x} \geq \overline{x}$, that is the expected trading probability is higher for low than for high type sellers. The remaining three conditions correspond to the properties that the mechanism m associated to $(\underline{x}, \overline{x}, \underline{t}, \bar{t})$ is feasible according to (1) and that meetings take place according to the urn-ball technology. In particular, inequality (9) requires that a buyer's probability of trading with a high type seller, $\overline{\lambda}\overline{x}$, is weakly smaller than a buyer's probability of meeting at least one high type seller, $\sum_{k=1}^{+\infty} P_k(\overline{\lambda}) = 1 - e^{-\overline{\lambda}}$. Inequality (10) is the analogous condition for the low type seller and inequality (11) requires that a buyer's probability of trading with any seller, $\overline{\lambda}\,\overline{x} + \underline{\lambda}\,\underline{x}$, is weakly smaller than the probability of meeting at least one seller, $1 - e^{-\overline{\lambda}-\underline{\lambda}}$.

Next, we rewrite the buyers' optimization problem in terms of an auxiliary problem, where a mechanism m, with the associated queue lengths $\overline{\lambda}, \underline{\lambda}$, is chosen so as to maximize a buyer's payoff, taking as given the utility gain attained in the market by low and high type sellers relative to their endowment points, denoted by $\underline{U}$ and $\overline{U}$ (in short, their market utilities). The associated queue lengths $\underline{\lambda}, \overline{\lambda}$ are constrained by conditions analogous to the sellers' optimality conditions in Definition 1 for the contracts posted in equilibrium and to conditions (5) and (6) restricting out of equilibrium beliefs. Such conditions can be viewed as a form of participation constraints.[20] This approach has been followed in earlier work in the literature on directed search (for instance, see Shi (2001 and 2002) for the case of public information and Eeckhout and Kircher (2010) for the case of independent private

[20] Letting the representative buyer optimize directly over arrival rates implies that in cases where there are multiple solutions to (5, 6), the buyer picks the preferred pair, which is consistent with condition iii) pinning down out of equilibrium beliefs. The auxiliary optimization problem will not allow the buyer to choose mechanisms for which the set of inequalities (5,6) does not have a solution. This comes without loss of generality because if $\underline{U}, \overline{U} > 0$, attracting infinitely many sellers always yields a strictly negative payoff and thus is never a solution of the auxiliary optimization problem.

values). While in these papers the optimization problem is formulated over the original space of mechanisms, in our set-up we rely on Proposition 3.2 to specify mechanisms simply in terms of values of expected trading probabilities and transfers satisfying the feasibility and incentive constraints (7)-(11). We have so the following optimization problem:

$$\max_{\underline{x},\overline{x},\underline{t},\overline{t},\overline{\lambda},\underline{\lambda}} \overline{\lambda}(\overline{x}\,\overline{v}-\overline{t})+\underline{\lambda}(\underline{x}\,\underline{v}-\underline{t}), \tag{P^{aux}}$$

subject to

$$\overline{t}-\overline{x}\,\overline{c} \quad \leq \quad \overline{U} \quad \text{holding with equality if } \overline{\lambda}>0, \tag{12}$$

$$\underline{t}-\underline{x}\,\underline{c} \quad \leq \quad \underline{U} \quad \text{holding with equality if } \underline{\lambda}>0, \tag{13}$$

and constraints (7)-(11).

It is immediate to verify that the solutions of the buyer's auxiliary problem as specified identify the mechanisms that are offered in equilibrium whenever the values of $\overline{\lambda}$ and $\underline{\lambda}$ are *consistent with the population parameters.* By consistent we mean that a probability measure β can be associated to the set of solutions $(\underline{x},\overline{x},\underline{t},\overline{t},\underline{\lambda},\overline{\lambda})$ of problem P^{aux} so that the feasibility condition (4) is satisfied. More specifically, if the solution to the auxiliary problem is unique, consistency simply requires that the optimal queue lenghts $\underline{\lambda}$ and $\overline{\lambda}$ coincide with the population parameters $\underline{\lambda}^p$ and $\overline{\lambda}^p$; in such case, there is an equilibrium where all buyers post the same mechanism. If the solution is not unique and the optimal queue lengths differ across the different solutions, consistency requires that the average value of queue lengths equals the population parameters, with weights equal to the fraction of buyers assigned to each solution; in such case, there is an equilibrium where sellers separate according to their type across different mechanisms.[21]

Hence finding an equilibrium amounts to finding values of the market utilities $\underline{U}$ and $\overline{U}$ such that the solutions of the buyer's auxiliary problem are consistent with the population. Since equilibria must be solutions of P^{aux}, the analysis of the general properties of these solutions allows us to establish two key properties: (i) equilibrium mechanisms will give priority to low type sellers and (ii) equilibria will not be separating.

[21]For example, suppose the buyer's auxiliary problem has two solutions with arrival rates, respectively, $\underline{\lambda}_1,\overline{\lambda}_1$ and $\underline{\lambda}_2,\overline{\lambda}_2$. If γ denotes the fraction of buyers posting in market 1, consistency requires $\gamma\underline{\lambda}_1+(1-\gamma)\underline{\lambda}_2=\underline{\lambda}^p$ and $\gamma\overline{\lambda}_1+(1-\gamma)\overline{\lambda}_2=\overline{\lambda}^p$.

(i) Priority for low type sellers. To establish this property we show that, at any solution of P^{aux}, we have $\underline{\lambda}\,\underline{x} = 1 - e^{-\underline{\lambda}}$. This condition says that a buyer's probability of trading a low quality good is equal to his probability of meeting a low type seller, which immediately implies property ii) of Theorem 3.1: any mechanism posted in equilibrium must give priority to low type sellers.[22] As already noted at the beginning of Section 3 this also implies that $\underline{x} > \overline{x}$. Thus no pooling mechanism, that treats equally the two types, is posted in equilibrium.

Lemma 3.3. *At any solution of P^{aux} the low type feasibility constraint (10) is satisfied with equality.*

Proof See Appendix A.3.

An important step for proving the above result is the observation that incentive compatibility of equilibrium mechanisms requires that $\underline{U} > \overline{U}$; that is, the expected utility gain by trading in the market, relative to the endowment, is strictly higher for low type sellers than for high type sellers (see Appendix A.2). Given this, the proof of Lemma 3.3 shows that whenever the posted mechanisms are such that low type sellers do not receive priority, buyers have a profitable deviation. This deviation is reminiscent of cream skimming and consists in offering a mechanism that attracts fewer low types and more high types, but gives priority to low type sellers so that the overall probability of trading a low and high quality good remains unchanged for the buyer. At the deviating mechanism sellers get the same utility as at the mechanisms posted in the market, but the buyer's profits are higher because the rents paid to sellers are strictly lower: since high type sellers obtain a strictly lower market utility than low type sellers, they are less costly to attract.

The above discussion illustrates how the priority rule proves an effective device for the principal to relax the incentive and participation constraints he faces, in particular of the low types, and to lower the transfer paid to these agents. It also shows that it is possible to do this without changing the probability a buyer trades with a high and a low quality seller, something the buyer cares for in an adverse selection environment (differently from the private value case).

(ii) No separating equilibria. We show next that we cannot have fully separating equilibria, where some buyers only attract high type sellers and others only attract low type

[22] This property has bite only for mechanisms that attract both low and high types. It still leaves open the possibility of a fully separating equilibrium where certain mechanisms only attract high type sellers.

sellers. For this to happen P^{aux} should have (at least) two solutions, one with $\underline{\lambda} = 0$ and the other with $\overline{\lambda} = 0$, and we prove that this is not possible.

Indeed the following lemma shows that the solution of P^{aux} with regard to the component of the mechanism concerning low type sellers, including their queue length, is always unique. This implies that in equilibrium there can be no posted mechanism that only attracts high types. The uniqueness of the complete solution of P^{aux} then depends on whether market utilities are such that buyers can or cannot make positive profits with high type sellers at the maximal level of their trading probability that is feasible and incentive compatible.[23]

Lemma 3.4. *(i) If market utilities are such that buyers can make strictly positive profits with high type sellers, there is a unique solution of P^{aux}.*

(ii) If market utilities are such that buyers can at most make zero profits with high type sellers, the solution of P^{aux} is only unique for $(\underline{\lambda}, \underline{x}, \underline{t})$.

Proof See Appendix A.4.

Lemma 3.4 shows that when there are mechanisms that satisfy the constraints of problem P^{aux} and yield the buyer strictly positive profits with high type sellers, the problem has a unique solution. Hence in equilibrium all buyers and sellers trade in a single submarket. To show uniqueness of the solution of P^{aux} in this case we proceed in two steps. First, we show that at any solution of P^{aux} the overall feasibility constraint (11) is satisfied with equality so that every meeting leads to trade. If not buyers would have a profitable deviation, which consists in attracting additional high type sellers, while keeping the terms of trade of all previous sellers unchanged. The rest of the proof of Lemma (3.4) then verifies that, given this property and the one established in Lemma 3.3, P^{aux} is a strictly convex problem.

When instead market utilities are such that buyers make at most zero profits with high type sellers, buyers are indifferent regarding the number of high type sellers they attract and there are so typically multiple values of $\overline{\lambda}$ that solve the buyer's auxiliary problem. Part (ii) of the lemma shows that in this case there is still a unique solution for the variables describing terms of trade and the queue length for low type sellers. The result follows again from the strict convexity of the choice problem with regard to these variables. It then follows that in equilibrium buyers could post different mechanisms, that however only differ in the terms of trade and the queue length for high type sellers.

[23]It can be shown (see Appendix A.2) that the maximal incentive feasible trading probability for high type sellers is $\overline{x}^{max} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. We thus distinguish the cases where $\underline{U}$ and $\overline{U}$ are such that the associated value of buyers' profits for such trades, $\overline{x}^{max}(\overline{v} - \overline{c}) - \overline{U}$, is strictly positive or zero.

To gain some intuition for why a fully separating equilibrium cannot exist, observe that in such an equilibrium high and low type sellers would trade in two separate markets, each of them generating the same profits for buyers. The latter property implies that buyers must be able to make strictly positive profits with high type sellers. Consider then a buyer who posts a mechanism which only attracts low type sellers. This buyer can profit by deviating to an alternative mechanism that also attracts high type sellers, but grants priority to low types and yields them the same trading probability and expected transfer as the original mechanism. Since buyers have no capacity constraints in their ability to meet sellers, these additional meetings do not crowd out any meetings with low type sellers. Hence, the alternative mechanism would attract the same number of low type sellers, and the buyer would obtain the same expected payoff from them; on top of that, the buyer can make some positive profits with high type sellers.[24]

Building on these properties, we are able to characterize in the next section the conditions under which the solutions of problem P^{aux} satisfy the consistency condition with the population parameters, thus establishing the existence of an equilibrium and some additional properties.

3.2 Existence of Equilibrium and Further Properties

3.2.1 Mild Adverse Selection

We begin by considering the case where market utilities are such that buyers can make strictly positive profits with high type sellers. Here we know from the analysis in the previous section that both the low type feasibility constraint (10) and the overall feasibility constraint (11) are binding at all solutions of P^{aux}, so we have:

$$\underline{\lambda}\,\underline{x} = 1 - e^{-\underline{\lambda}} \quad \text{and} \quad \overline{\lambda}\,\overline{x} = e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right). \tag{14}$$

Since, as shown in Lemma 3.4(i), the solution of problem P^{aux} is unique, to obtain an equilibrium we need to find values of $\underline{U}$ and $\overline{U}$ such that the population ratios $\underline{\lambda}^p$ and $\overline{\lambda}^p$, together with the associated values of $\underline{x}, \overline{x}$ obtained from (14), solve problem P^{aux} and profits

[24]If buyers can offer high type sellers the same terms of trade as in the market where only high types trade, each additional high type generates the same profit as in this market and the profitability of the deviation follows immediately. Sometimes providing the same terms of trade will not be feasible because the priority for low type sellers imposes constraints on the trading probability of high type sellers. Hence transfers must be adjusted to attract high types, but our proof shows that also in this case additional profits can be generated.

with high type sellers are indeed positive. The following proposition identifies a condition on the parameter values of the economy under which such values of $\underline{U}, \overline{U}$ exist. The argument of the proof also allows to determine the equilibrium values of $\underline{U}, \overline{U}$, thereby characterizing the equilibrium allocation in closed form.

Proposition 3.5. *There exists a competitive search equilibrium in which buyers make strictly positive profits with high type sellers and post the same mechanism, with trading probabilities*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right),$$

if and only if $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.

Proof See Appendix A.5.

Condition $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is always satisfied in the case of independent private values ($\underline{v} = \overline{v}$), and holds more generally when buyers do not care much for quality ($\overline{v} - \underline{v}$ is sufficiently small) and the competition for high type sellers is not intense (their ratio to buyers $\overline{\lambda}^p$ is sufficiently large).[25] In this situation in equilibrium the utility gain of high type sellers compared to that of low type sellers is sufficiently small so that buyers make strictly positive profits with high type sellers and all meetings lead to trade. We refer to this parameter region as characterizing a situation where adverse selection is 'mild'.[26]

3.2.2 Severe Adverse Selection

We turn next our attention to the alternative scenario where market utilities are such that buyers can make at most zero profits with high type sellers. In this case buyers that attract high type sellers will offer a mechanism at which the trading probability of high type sellers is at the highest possible level that is feasible and incentive compatible.[27] The trading probability of low type sellers is again determined by the priority rule and thus specified by (14). We then proceed, similarly to the previous section, to identify conditions on the

[25] Notice that the function $f(\lambda) = \frac{1}{\lambda}(1 - e^{-\lambda})$ is strictly decreasing and lies between zero and one on its domain $(0, +\infty)$: we have in fact $f'(\lambda) = -\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda^2} < 0$ for all $\lambda > 0$, $\lim_{\lambda \to 0} f(\lambda) = 1$ and $\lim_{\lambda \to +\infty} f(\lambda) = 0$.

[26] The property that a single mechanism is posted in equilibrium is also present in the results obtained in the literature when agents' types are publicly observable or with independent private values. Here however it does not follow from the efficiency property of the equilibrium. Moreover, as we show next, with severe adverse selection, the equilibrium may also feature partial separation.

[27] That is, $\bar{x} = \overline{x}^{max} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ (see footnote 23).

parameter values under which the solution of P^{aux} yields an equilibrium. In this case, as we know from Lemma 3.4(ii), problem P^{aux} has multiple solutions for $\overline{\lambda}$. We have so multiple, payoff equivalent equilibria, as the following proposition shows.

Proposition 3.6. *There exists a set of payoff equivalent competitive search equilibria, in each of which buyers make zero profits with high type sellers and post mechanisms such that*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}},$$

if and only if $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. *Whenever the inequality is strict the set has a continuum of equilibria; it always includes one where all buyers post the same mechanism.*

Proof See Appendix A.6.

The condition under which the above equilibria exist, $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, is the exact complement of the one stated in Proposition 3.5. Note that this inequality can only be satisfied if $\overline{v} > \underline{v}$, that is in a situation of common values. More precisely, the condition requires that buyers care a lot for quality ($\overline{v} - \underline{v}$ sufficiently large) and that there are relatively few high type sellers per buyer ($\overline{\lambda}^p$ sufficiently small). We refer to such situation as one of 'severe' adverse selection. The stated properties imply that now there is intense competition for high type sellers. This leads buyers to make zero profits in equilibrium in trades with high types, who extract so all the gains from trade that are realized. Hence, in contrast to results obtained in cited literature on competitive equilibria with adverse selection, buyers' profits are not equalized across trades with the two types.

The feature that buyers make zero profits with high type sellers is closely linked to two other important equilibrium properties, which we discuss next.

Rationing. If condition $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is satisfied with strict inequality, at the equilibrium where all buyers post the same mechanism the probability that a buyer ends up trading a high quality object, given by $\overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, is strictly smaller than the probability that a buyer is in a match with high type sellers only, $e^{-\underline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})$. Thus not all meetings with only high types end up with trade.

To gain some intuition for why rationing occurs in equilibrium, it is useful to observe that in equilibrium the payoff of a buyer conditional on meeting a low type seller must be strictly higher than his payoff conditional on meeting a high type seller, i.e.

$$\underline{x}(\underline{v} - \underline{c}) - \underline{U} > \overline{x}(\overline{v} - \overline{c}) - \overline{U}. \tag{15}$$

If, on the contrary, buyers would make more profits with high type sellers, a profitable cream skimming deviation would exist. Since, as we showed, sellers with a high quality good trade with a lower probability than sellers with a low quality good, replacing low type sellers with high type sellers while keeping their trading probabilities and transfers unchanged would be feasible and profitable.[28]

Property (15) holds for all parameter values. When gains from trade of the high quality good are large compared to those of the low quality good, it implies that most of these gains are appropriated by the sellers, which means that the high quality object must be traded at a high price. Incentive compatibility then requires that the trading probability of high type sellers is sufficiently small. Giving priority to low type sellers may not be sufficient, particularly when the ratio of high type sellers to buyers is sufficiently small, to guarantee a small enough trading probability and so high type sellers must be rationed. Crucially, the larger the buyers' valuation of the high quality good is, the more severe rationing becomes, which can be seen formally by observing that the high type sellers' equilibrium trading probability decreases in $\overline{v}$ (see Proposition 3.6). This feature stands in contrast to a monopolistic auction setting, or one with random rather than directed search, where larger values of $\overline{v}$ favour the posting of pooling mechanisms with the two types trading with the same probability and thus leads to a weakly larger trading probability of the high quality good.[29] Our environment, with competing buyers posting general mechanisms delivers quite the opposite comparative statics result.

To gain some intuition for this fact we should point out that both in the monopolistic auction and the random search setting, the buyers faces a set of sellers, with given proportions among the different types. In our set-up each buyer faces competition from other buyers and so the composition of the sellers he attracts depends on the mechanism he posts: in particular, if he were to post a pooling price, thereby offering a higher information rent for low type

[28] A similar argument applies if buyers make the same profits with high and low type sellers. The profitable deviation in this case consists in attracting a slightly larger number of high types, which is again feasible due to $\underline{x} > \overline{x}$.

[29] See Manelli and Vincent (1995) for the case of a single buyer who can purchase the good from multiple sellers and Koeppl and Chiu (2016) and Maurin (2017) for the case of random search. In the latter case each buyer meets a single seller and optimally posts a price (see Samuelson, 1984)

sellers, he would only attract low types.[30]

Partial Sorting When $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ holds, Proposition 3.6 shows we can sustain multiple, payoff equivalent equilibria that differ in their assignment of high type sellers across buyers. In particular, rather than having each buyer attract the same average number of high type sellers, we can distribute high type sellers asymmetrically across submarkets.

To illustrate this, consider the following specification with two active submarkets, labelled 1 and 2. The respective seller-buyer ratios are $\overline{\lambda}_1, \underline{\lambda}_1$ and $\overline{\lambda}_2, \underline{\lambda}_2$. For simplicity set $\overline{\lambda}_2 = 0$ so that high type sellers only visit submarket 1. The trading probabilities for low type sellers in the two submarkets and for high type sellers in submarket 1 are then as specified in Proposition 3.6.[31] By Lemma 3.4 both $\underline{\lambda}_1$ and $\underline{\lambda}_2$ must be equal to $\underline{\lambda}^p$. Let γ denote the fraction of buyers posting mechanism 1. Consistency with the population parameters requires that $\overline{\lambda}_1 = \frac{\mu s}{\gamma b} = \frac{1}{\gamma}\overline{\lambda}^p$, that is, the queue length of high type sellers in the first submarket, $\overline{\lambda}_1$, is equal to the ratio between all high type sellers in the economy, μs, and those buyers that visit the first submarket, γb. Furthermore, the value of $\overline{\lambda}_1$ must be such that the overall feasibility constraint (11) is satisfied, which, after substituting the values specified above, requires[32]

$$\frac{1}{\frac{1}{\gamma}\overline{\lambda}^p}\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}. \tag{16}$$

Under the stated parameter restriction, the above condition is always satisfied for an interval of values of γ sufficiently close to one, that is, such that sufficiently many buyers and low type sellers participate in submarket 1, the one where both types of objects are traded.

In the situation described, two mechanisms coexist in equilibrium. In submarket 2 buyers post a simple mechanism (effectively a price) that only attracts low type sellers, while in submarket 1 buyers post a more complex mechanism (some form of auction) attracting both high and low type sellers. Notice that γ can be set so that the overall feasibility constraint (16) is satisfied with equality, in which case there is no rationing in either market.

These two properties of equilibria with severe adverse selection show that the reduction

[30] Some further explanation of this property can be found in Section 4 where we discuss the welfare properties of competitive search equilibria.

[31] In contrast, in submarket 2 where $\overline{\lambda}_2 = 0$ the value of high types' trading probability can be chosen freely as long as the low type incentive constraint is satisfied, for instance, it could be equal to that of the low types.

[32] Letting $\overline{x}_1$ denote the trading probability of high types in submarket 1, we have: $\overline{\lambda}_1\overline{x}_1 = \frac{1}{\gamma}\overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} \leq e^{-\underline{\lambda}^p}\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right)$, which yields the expression in the text.

of the equilibrium trading probability of high type sellers, needed to satisfy incentive compatibility for low type sellers, can be achieved in different ways: either by rationing high types within the mechanism, or by assigning high types asymmetrically across buyers, or by a combination of the two.

Remark: *The parameter region with severe adverse selection includes as a limiting case the situation where there is free entry of buyers, captured in our environment by letting the measure of buyers tend to $+\infty$. This case is of interest since it corresponds to the situation considered in Rothschild and Stiglitz's (1976) classic model of adverse selection as well as in other models where there is free entry of uninformed traders (or equivalently, each uninformed trader can trade any number of contracts). As $b \to +\infty$, the ratio of high type sellers to buyers, $\overline{\lambda}^p$, tends to zero, implying that $\frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})$ tends to one. From the proof of Proposition 3.6 it can be seen that, as $b \to +\infty$, the transfer to low type sellers conditional on trading, $\underline{t}/\underline{x}$, converges to $\underline{v}$, while their trading probability converges to one; the transfer to high type sellers conditional on trading, on the other hand, is equal to $\overline{v}$, while their trading probability converges to $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. These values precisely correspond to the ones of the separating candidate equilibrium found by Rothschild and Stiglitz (1976).*[33]

3.3 Implementation

Propositions 3.5 and 3.6 characterize the values of the expected trading probabilities of the equilibrium mechanisms buyers post. In their proofs we also determine the associated market utilities which yield the expected transfer payments for these mechanisms. What can we say regarding the properties of trading rules that implement these values? As pointed out in the introduction, for the case of independent private values earlier work (see in particular Peters, 1997) has shown the existence of a competitive search equilibrium where all buyers post the same mechanism and the equilibrium trading probabilities and transfers can be implemented through a second-price auction with a reserve price equal to the buyers' valuation.[34]

Our equilibrium characterization shows that, in the case of mild adverse selection trading probabilities, as determined by the priority for low types and the fact that all meetings of a

[33]Note that in competitive search models, as well as in Walrasian models, the non-existence issue found by Rothschild and Stiglitz (1976) in a strategic setting does not arise.

[34]It should be emphasized that, compared to us, some of these papers consider more general specifications of the type space (e.g. Peters, 1997) or of the the meeting technology (e.g. Eeckhout and Kircher, 2010) than in our set-up.

buyer with some sellers end up with trade, are the same as in a second price auction. The associated transfers however can never be implemented through a standard second-price auction with reserve price r, not even if coupled with a participation fee or transfer κ. To see this, notice that the payoffs of low and high type sellers associated to such an auction, denoted by $m_{r,\kappa}$, with associated seller-buyer ratios $\underline{\lambda}^p, \overline{\lambda}^p$, are given by[35]

$$\begin{aligned}\underline{u}(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p}\left(e^{-\overline{\lambda}^p}(r-\underline{c})+(1-e^{-\overline{\lambda}^p})(\overline{c}-\underline{c})\right)+\kappa,\\ \overline{u}(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p-\overline{\lambda}^p}(r-\overline{c})+\kappa.\end{aligned}$$

Setting these values equal to the equilibrium market utilities we obtained yields two linearly dependent equations in r and κ. As can be verified, these equations have no solution, except when $\underline{v}=\overline{v}$ (that is, in the case of independent private values).

In contrast, the equilibrium trading probabilities and transfers can be implemented, for instance, via a sequential auction. Buyers could first run a second-price auction with reserve price $r_1 \in (\underline{c},\overline{c})$ and, if nobody wins the auction, run another auction with reserve price $r_2 > \overline{c}$ with probability one (when adverse selection is mild), or less than one (when it is severe). It is clear that such sequential auction generates the trading probabilities stated in Propositions 3.5 and 3.6. Furthermore, it can be easily verified that there always exists a pair (r_1, r_2) (possibly coupled with a participation transfer or fee κ) such that the sellers' expected utilities associated to this mechanism are equal to the equilibrium market utilities specified in the proofs of Propositions 3.5 and 3.6.[36]

[35] In such a second price auction a low type seller earn a positive payoff if and only if he is the only low type seller in a meeting, an event with probability $e^{-\underline{\lambda}^p}$ (his actual payoff then differs according to whether there are, or there are not, also high type sellers). For a high type seller a positive payoff arises only if he is the only seller in a meeting (with probability $e^{-\underline{\lambda}^p-\overline{\lambda}^p}$).

[36] Letting $m_{r_1,r_2,\kappa,\rho}$ denote the mechanism with sequential reserve prices r_1, r_2, participation transfer κ, and a conditional probability of going to the second round ρ, the low and high type seller's payoff are, respectively, given by

$$\begin{aligned}\underline{u}(m_{r_1,r_2,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p-\overline{\lambda}^p}(r_1-\underline{c})+\kappa,\\ \overline{u}(m_{r_1,r_2,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p-\overline{\lambda}^p}\rho(r_2-\overline{c})+\kappa.\end{aligned}$$

4 Welfare

In the economy under consideration the level of total surplus coincides with the realized gains from trade. At an allocation where the trading probabilities are, respectively, $\overline{x}$ and $\underline{x}$ for the high and low type sellers it is then given by

$$b\left[\overline{\lambda}^p\overline{x}(\overline{v}-\overline{c})+\underline{\lambda}^p\underline{x}(\underline{v}-\underline{c})\right].$$

The welfare properties of the equilibria we characterized depend on the values of the parameters of the economy. We first show that, when adverse selection is mild and gains from trade are larger for the low quality object, maximal surplus is attained in equilibrium:

Proposition 4.1. *At the competitive search equilibrium allocation, total surplus is maximal, relative to all feasible allocations, if*

$$\underline{v}-\underline{c}\geq\overline{v}-\overline{c}\quad\textit{and}\quad\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)\leq\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

If either of the two conditions is violated, there exists a feasible and incentive compatible allocation that generates strictly more total surplus. Moreover, if μ is sufficiently large, this allocation constitutes a Pareto improvement with respect to the one of the competitive search equilibrium.

Proof See Appendix A.7.

In the first part of the proposition we consider the set of allocations that are attainable subject only to the meeting friction, that is ignoring incentive compatibility. In this set, total surplus is maximal if the number of meetings is maximized, every meeting leads to trade and sellers owning the good with the larger gains from trade receive priority. It is well known that under the urn-ball meeting technology the total number of meetings is maximal whenever the queue length of sellers is the same for all the mechanisms traded in equilibrium.[37] The restriction $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)\leq\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ assures that in equilibrium all sellers

[37]See for example Eeckout and Kircher (2010). For the private value case Cai et al. (2017a) in fact demonstrate that the merging of any two submarkets increases total surplus under any type distribution if and only if the meeting technology satisfies a property called 'joint concavity'. According to this property, the probability of meeting at least one low type seller is a concave function of the queue length of low type sellers and the queue length of high type sellers. For completeness of the argument we provide a formal proof of the property that in our environment the number of meetings is maximized when buyers and sellers trade in a single submarket in the Online Appendix.

trade in a single submarket and that high type sellers are not rationed in equilibrium, so the first two properties stated above are indeed satisfied. When $\underline{v} - \underline{c} \geq \overline{v} - \overline{c}$ the sellers with the larger gains from trade are those with a low quality good, and so the property that in equilibrium low types sellers have priority in trades implies that in this case also the last requirement is satisfied.

Notice that the two conditions stated in Proposition 4.1 are always satisfied in the case of independent private values, $\underline{v} = \overline{v}$. The result that the competitive search equilibrium maximizes total surplus in private value environments is well established in the literature (see for example Eeckhout and Kircher, 2010, and Cai et al., 2017). The first part of Proposition 4 extends this result to the case of common values, provided gains from trade are still larger for the low quality object and the ratio of high type sellers to buyers is not too low. In such a situation incentive compatibility and adverse selection do not constrain attainable welfare.

When instead buyers care sufficiently for quality, so that the gains from trade of the low quality object are strictly smaller than those of the high quality object, or when there are sufficiently few high type sellers so that they are rationed in equilibrium, the welfare properties of equilibria are quite different. The second part of the proposition shows that in this case the allocation of the competitive search equilibrium no longer maximizes total surplus, even subject to the additional constraint imposed by incentive compatibility. Moreover, if the fraction μ of high type sellers relative to low type sellers is sufficiently large, there exists an allocation that satisfies the constraints imposed by the meeting friction and incentive compatibility and Pareto improves on the allocation of the competitive search equilibrium, that is the equilibrium is not incentive constrained efficient.

More specifically, the proof of this last part of the claim shows that, when either of two stated conditions is violated, an increase in the trading probability of high type sellers relative to their equilibrium level, possibly combined with a reduction of the low types' trading probability and a suitable adjustment of the transfers, is both feasible and incentive compatible. This change in the allocation increases total surplus and improves sellers' utility. It is then shown it also constitutes a Pareto improvement, in the sense that buyers gain as well, provided the fraction of high relative to low type sellers is sufficiently high. In this case the additional gains made by buyers with high type sellers more than compensate the possible losses with low types.

The reason why at the competitive search equilibrium characterized in Section 3 there is no profitable deviation that allows to capture the additional gains from trade described in the previous paragraph is that such deviation would attract too many low type sellers in order to be profitable. To see this, notice that, as shown in the proof of the second part of Proposition 4, in order to increase the trading probability of high type sellers a buyer would have to pay an additional information rent to low type sellers. This implies that all low type sellers would have strict incentives to switch to the deviating contract, thereby making the deviation non profitable.[38]

4.1 General Versus Bilateral Mechanisms

The inefficiency of competitive equilibria with adverse selection we found should not surprise us. Analogous inefficiency results were obtained in the literature with bilateral contracting, both for competitive equilibria with directed search when meetings are restricted to be bilateral (see Gale (1992) and Guerrieri et al. (2010)) and for Walrasian equilibria without search frictions (Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006)). In all these cases the forces of competition among (uninformed) buyers prevent to internalize the externalities induced by incentive compatibility linking the different types of (informed) sellers. The properties of the allocations obtained in equilibrium are however quite different. As mentioned in the introduction, a common feature with bilateral contracting is that equilibria are separating, with different types of sellers trading different contracts and buyers' profits equalized across the trades with each type.[39] In contrast, in our set-up all sellers are pooled in the same mechanism and profits are not equalized across types. The fact that inefficiency may still obtain suggests that equilibrium and the competition among buyers impose some conditions on the directions of the cross-subsidies, which may go against what would be required to achieve efficiency. Given the rather different features of the equilibrium allocation in the two set-ups, the question arises how they compare in terms of welfare.

To conduct such a comparison within our framework we consider next the case where the space of available mechanisms is restricted to the class of bilateral menus, while keeping the meeting technology unchanged. Such restriction entails that a buyer picks at random one of the arriving sellers, who can then choose his preferred item from the menu that is proposed.

[38]More precisely, if a buyer posts a mechanism m that yields a higher trading probability for high types, incentive compatibility requires that he offers a payoff for low type sellers strictly larger than $\underline{U}$ for any pair $\underline{\lambda}, \overline{\lambda}$ (see the proof of Proposition 4.1). The buyer's belief is thus pinned down by condition (ii) of Definition 1: $\underline{\lambda}(m) = +\infty$. Given this belief, the deviating contract yields a strictly negative payoff for the buyer.

[39]The 'no cross-subsidization' property is a key factor for the inefficiency that arises in such markets. As Davoodalhosseini (2017) shows, if adverse selection is sufficiently severe, a planner can improve on the competitive search equilibrium allocation by taxing the high quality market and subsidizing the low quality market. The cross-subsidization relaxes the incentive compatibility constraint of low type sellers and thereby allows for more trades of the high quality good.

Conditionally on being chosen, a seller's trading probabilities and transfers thus depend on his own report but not on the report of others.

Under urn-ball matching the probability for a seller to be chosen at random from those sellers arriving at the buyer when the overall queue length is $\underline{\lambda}+\overline{\lambda}$ is given by $\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$. Clearly, the trading probabilities of bilateral menus must satisfy the constraints $\overline{x},\underline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$. Moreover, trading probabilities and transfers must not condition on the reports of sellers other than the one selected by the buyer. When deriving the competitive search equilibrium we will ignore the latter restriction and verify that, with the constraints $\overline{x},\underline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$ in place, the set of solutions of the buyers' auxiliary optimization problem are such that trading probabilities and transfers are indeed independent of the reports of other sellers arriving at the same buyer.

The next proposition shows that, given the restriction to bilateral menus, the equilibrium outcome features different mechanisms posted in equilibrium. These mechanisms treat all types the same ($\overline{x}=\underline{x}$) and do not ration any sellers. Hence, rather than posting menus, buyers offer single contracts, which take the form of posted prices. Sellers separate themselves according to their type across the different prices posted and buyers' profits are equalized across trades, in line with the results in the literature on bilateral contracting recalled above.

Proposition 4.2. *If the set of available mechanism is restricted to bilateral menus, a competitive search equilibrium exists and has the following properties:*

- *a fraction $\gamma \in [0,1)$ of buyers post a price p_h and only attract high type sellers;*
- *the remaining fraction $1-\gamma \in (0,1]$ of buyers post a price $p_l < p_h$ and only attract low type sellers.*

Proof See the Online Appendix.

As we show in the proof of the proposition, under the stated restriction on the feasible mechanisms, buyers never find it optimal to attract both types of sellers. Thus separation no longer occurs within a mechanism, though this is still possible, but with different mechanisms being posted.

We can now compare, in terms of welfare, the equilibrium allocations with bilateral contracts to those with general mechanisms. We will see that this comparison yields distinct and surprising results in the case of adverse selection, as compared to the situation with independent private values or public information. Recall first that, if the gains from trade for the low quality good exceed those for the high quality good and the ratio of high type sellers is not too small, the equilibrium with general mechanisms maximizes total surplus (see Proposition 4.1). Hence it always dominates the one with bilateral contracts in terms of total surplus. The interesting case to consider is the one where the gains from trade are larger for the high quality good, as the equilibrium with general mechanisms may be incentive constrained inefficient. As we saw earlier, the inefficiency lies in the fact that high types trade too little. Their level of trade is particularly low when adverse selection is severe: in that case, not only the mechanism traded in equilibrium gives priority to the good with the lower gains from trade but there also can be rationing on top of the meeting friction. One might conjecture that the equilibrium with bilateral contracts may do better in such situations. However, we find that this is typically not the case, as the next numerical example illustrates.[40]

Example 1: *Let* $\underline{\lambda}^p = \overline{\lambda}^p = 1$ *and* $\underline{c} = 0, \overline{c} = 1, \underline{v} = 1, \overline{v} = 3$. *Thus the gains from trade for the high quality good are twice as large as those for the low quality good and there are twice as many buyers as sellers, half of whom have a high quality good. Under this specification, we have* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, *that is adverse selection is severe. The equilibrium with general mechanisms where all buyers post the same mechanism is then as characterized in Proposition 3.6 and features rationing. In particular, a buyer's probability to trade, respectively, a high and low quality good is given by*

$$\begin{aligned} \overline{\lambda}^p \overline{x} &= \overline{\lambda}^p e^{-\underline{\lambda}^p} \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}} \approx 0.123, \\ \underline{\lambda}^p \underline{x} &= 1 - e^{-\underline{\lambda}^p} \approx 0.632. \end{aligned}$$

while a buyer's probability to meet some high type seller without meeting a low type seller is $e^{-\underline{\lambda}^p}(1 - e^{-\overline{\lambda}^p}) \approx 0.233$. *Hence, in meetings without low type sellers trade occurs only slightly more than half of the time.*

In the equilibrium where mechanisms are restricted to bilateral menus a fraction $\gamma \approx 0.120$ *of buyers post the high price, while the remaining fraction of buyers post a low price. A buyer's*

[40] We considered several other specifications of the parameters in the region with severe adverse selection. In all the cases we found that total surplus in the equilibrium with general mechanisms exceeds that in the equilibrium with bilateral contracts.

probability to trade a high and low quality good is now

$$\begin{aligned}\overline{\lambda}^p\overline{x} &= \gamma\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right)\approx 0.120,\\ \underline{\lambda}^p\underline{x} &= (1-\gamma)\left(1-e^{-\frac{1}{1-\gamma}\underline{\lambda}^p}\right)\approx 0.598.\end{aligned}$$

In the above example we see that in the equilibrium with bilateral contracts the probability that a buyer meets a seller decreases, when compared to the equilibrium with general mechanisms, from 86.5% to 71.8%. This is due to the properties of the meeting technology: as already noticed, allocating buyers and sellers over two submarkets with different seller-buyer ratios implies that there is a higher chance that sellers end up misallocated across buyers. The probability of trading a low quality good decreases as well and this is also intuitive, since with bilateral contracts low type sellers distribute themselves only across a fraction of buyers rather than across all of them.

What is perhaps more surprising is that, in the situation considered in Example 1, in the equilibrium with bilateral contracts also the probability of trading a high quality good is slightly lower, 12.0% instead of 12.3%, which implies that total surplus is unambiguously lower. To gain some understanding for why this happens, notice that in such an equilibrium, for low type sellers to choose to trade at the lower price p_l the seller-buyer ratio in the high quality market must be higher than in the low quality market. This implies that a buyer's probability of meeting a high type seller strictly exceeds that of meeting a low type seller. Since in equilibrium buyers have to be indifferent between attracting high and low type sellers, it follows that, conditional on meeting a seller, a buyer has to make lower profits with high than with low type sellers. This in turn implies that most of the gains from trade of the high quality good have to go to high type sellers. Hence incentive compatibility is only satisfied if a high type seller's trading probability is sufficiently low, the more so the higher these gains from trade are, similarly to the case where buyers post general mechanisms.

At the same time, it is important to point out that total surplus is not always higher in the equilibrium with general mechanisms. Notably, this reversal arises when parameters fall in the region where in such equilibrium every meeting leads to trade. The next proposition demonstrates that, provided the gains from trade for the high quality good exceed those for the low quality good and the measure of high type sellers is sufficiently large (so that competition among buyers for high type sellers is limited), the equilibrium with bilateral contracts yields a strictly larger total surplus compared to the equilibrium with general mechanisms.

Proposition 4.3. *Assume $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. If the measure of high type sellers is sufficiently large, total surplus is strictly greater when buyers are restricted to bilateral menus compared to when they can post general mechanisms.*

Proof See Appendix A.8.

The result is established by considering the properties of the equilibrium allocations as $\overline{\lambda}^p \to +\infty$ and can be explained as follows. The conditions $\underline{v} - \underline{c} < \overline{v} - \overline{c}$ and $\overline{\lambda}^p$ sufficiently high imply that we are in a situation of mild adverse selection and the equilibrium with general mechanisms features no rationing. The property that low type sellers are given priority in every meeting implies that buyers trade the low quality good with probability $1 - e^{-\underline{\lambda}^p}$, the probability with which they meet at least one low type seller. As $\overline{\lambda}^p \to +\infty$, a buyer's probability of meeting some high type seller tends to one which implies, since there is no rationing, that in the limit buyers trade a high quality good with the residual probability, $e^{-\underline{\lambda}^p}$. Total surplus thus tends to $b[e^{-\underline{\lambda}^p}(\overline{v} - \overline{c}) + (1 - e^{-\underline{\lambda}^p})(\underline{v} - \underline{c})]$.

In the equilibrium with bilateral contracts total surplus is strictly higher because, as $\overline{\lambda}^p \to +\infty$, the fraction γ of buyers attracting high type sellers tends to one. In this equilibrium the trading probability of sellers converges to zero in both submarkets, but the relative probability of trade in the high quality submarket compared to the low quality submarket is sufficiently small so that the incentive compatibility constraint of low type sellers is satisfied. As a consequence, the probability that a buyer trades tends to one in both submarkets and the measure of buyers posting the high price tends to b. In the limit, total surplus in the equilibrium with bilateral contracts is thus given by $b(\overline{v} - \overline{c})$, which is equal to the first best level, as all buyers purchase the high quality object. This value strictly exceeds total surplus in the equilibrium with general mechanisms, where, due to the property that low type sellers are given priority over high type sellers, a positive fraction of buyers ends up purchasing the low quality object. The difference in total surplus between the two types of equilibria is largest when also the measure of low type sellers is large. These properties are also illustrated in the following example.

Example 2: *Let $\overline{\lambda}^p = 8$, $\underline{\lambda}^p = 2$ and $\underline{c} = 0, \overline{c} < 1.5, \underline{v} = 2.5, \overline{v} = 4$. Again the gains from trade for the high quality good are strictly greater than those for the low quality good. However, compared to Example 1, for every buyer there are eight high type sellers and two low type sellers. Under this specification, we have $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}$. In the equilibrium* *with general mechanisms there is no rationing and a buyer's probability to trade, respectively, a high and a low quality good is given by*

$$\begin{aligned}\overline{\lambda}^p \overline{x} &= e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \approx 0.135,\\ \underline{\lambda}^p \underline{x} &= 1-e^{-\underline{\lambda}^p} \approx 0.865.\end{aligned}$$

In the equilibrium where mechanisms are restricted to bilateral menus the fraction of buyers posting the high price is $\gamma \approx 0.392$ *and the probability a buyer trades, respectively a high and low quality good is given by*

$$\begin{aligned}\overline{\lambda}^p \overline{x} &= \gamma\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \approx 0.392,\\ \underline{\lambda}^p \underline{x} &= (1-\gamma)\left(1-e^{-\frac{1}{1-\gamma}\underline{\lambda}^p}\right) \approx 0.585.\end{aligned}$$

In Example 2 the equilibrium with general mechanisms has the feature that, due to the high seller-buyer ratio, almost all buyers are matched. However only 13.5% of them end up purchasing a high quality good, since there are twice as many low type sellers as buyers and so the probability that a buyer meets some low type seller is relatively high (86.5%). Thus, although the majority of sellers have a high quality good, the feature that low type sellers are given priority in any match, together with a large seller-buyer ratio, implies that high quality is traded relatively rarely.

In the equilibrium with bilateral contracts, on the other hand, the probability that a buyer trades is slightly lower (97, 7%) but the probability of trading a high quality good is considerably higher (almost 40%). Whether this leads to an increase in surplus or not depends on the seller's valuation of the high quality good. If $\overline{c}$ is sufficiently small (in our numerical example $\overline{c} < 1.28$), so that sellers do not care much for quality, the effect of the increased probability of trade of the high quality object outweighs the effect of the reduced overall probability of trade and surplus is larger in the equilibrium with bilateral contracts.

To sum up, the analysis in this section shows that the features of the trading mechanisms between buyers and sellers that are available matter considerably. While for most parameter specifications the equilibrium with general mechanisms yields a higher level of total surplus than the equilibrium where mechanisms are restricted to bilateral contracts, this is not always the case. Hence, there are some situations in which policies imposing restrictions on the set of available mechanisms are beneficial but in several other situations improving policies are those that encourage meeting technologies without capacity constraints, with no restrictions on the set of mechanisms.[41]

5 Conclusion

The paper shows that allowing uninformed buyers to post general mechanisms, instead of bilateral contracts, in markets with adverse selection has important implications for the equilibrium allocation and its welfare properties. In particular, we find that there always exists a competitive search equilibrium where all buyers post the same mechanism that gives priority to sellers with a low quality good. Hence separation of the different types of sellers occurs within the same mechanism that is posted, where high type sellers only get to trade in meetings without low type sellers, no matter how large are the gains from trade of the good they own. This property of the equilibrium can give rise to severe inefficiencies, which in some situations call for regulatory measures limiting buyers' choice of mechanism.

To keep the analysis tractable we restricted attention to the case where sellers' types are binary. In what follows we provide some indication of how our results may extend to an environment with $N \geq 3$ types. Observe first that the argument for why low types must receive priority over high types does not rely on the fact that there are only two types of sellers (see the paragraph following Lemma 3.3). The fact that cream skimming deviations must not be profitable in equilibrium implies that sellers only get to trade in meetings where no seller of lower type is present. It equally implies that buyers' profits conditionally on meeting a seller must be decreasing in the seller's type, even when several types exist. It remains then to be seen whether the trading probabilities associated to such priority mechanism can be combined with a schedule of transfers such that buyers' profits are indeed decreasing in the sellers' type and incentive compatibility is still satisfied. If this is not the case, we conjecture that the equilibrium mechanism rations some types of sellers on top of what the priority rule implies. In such situations, by arguments similar to that of Lemma 3.4,[42] buyers must make zero profits with all types of sellers that are rationed in equilibrium. This implies that buyers make zero profits also with sellers of any higher type. Thus, in an environment with $N \geq 3$ types of sellers, we conjecture that buyers trade according to a priority mechanism and there

[41]Lester et al. (2017a) study a related issue in an environment with random search and imperfect competition. In particular, they examine how the features of the meeting technology affect traders' market power and hence the consequences for the welfare properties of equilibria in the presence of adverse selection.

[42]Lemma 3.4 shows that whenever buyers can make positive profits with high type sellers, a mechanism that rations high type sellers cannot be optimal for a buyer. Viceversa, one can show that, if the optimal mechanism rations a certain type of seller, buyers must make at most zero profits with that type.

might be a cutoff type such that buyers make strictly positive profits with seller types below the cutoff and zero profits with the remaining sellers, who are rationed in equilibrium.

Our analysis is further based on the consideration of a specific matching technology and trading protocol. In particular, we assumed that mechanisms are posted by buyers, the uninformed party, who are not constrained in their ability to meet sellers and that sellers can at most meet one buyer. While we believe these features provide a natural benchmark to analyse the effects of allowing for general trading mechanisms, instead of bilateral contracts, the consideration of environments that differ in some of these features raises some interesting issues. In ongoing work we have been considering in particular the case of more general matching technologies where buyers face capacity constraints in their ability to meet sellers.[43] The presence of these constraints implies that the deviations, that we saw prevented the existence of fully separating equilibria, now entail a cost (attracting high types requires attracing fewer low types) and so may no longer be profitable.

Some interesting issues also arise if sellers were allowed to meet more than one buyer, and hence participate in more than one mechanism. This amounts to dropping the maintained assumption of exclusivity and implies that sellers may end up getting at the same time more than a single opportunity to trade, possibly at different terms, so that the offers generated by the mechanism proposed by a buyer may end up being rejected. Taking that into account requires buyers to appropriately modify the terms of the mechanisms posted and also changes the nature of competition among them (see Galenianos and Kircher (2009) and Wolthoff (2017) for some first steps in this direction in environments with complete information).

The property that buyers post mechanisms is fairly standard as it allows to avoid issues of signaling and the associated multiplicity of equilibria. Furthermore, as already argued in Section 2, in environments where the heterogeneity is one-sided allowing for general mechanisms rather than bilateral contracts only makes a difference when the individuals involved in the mechanism are heterogeneous. Extending the analysis to markets with two-sided heterogeneity is clearly of interest. Delacroix and Shi (2013), as well as Albrecht et al. (2016), provide interesting work exploring the consequences of allowing for signaling in directed search models, though in environments rather different from ours.[44]

[43]This line of research has been pursued for the case of independent private values by Cai et al. (2017a and 2017b).

[44]In Delacroix and Shi (2013) sellers are ex-ante identical but decide which type of product to produce, hence the environment features moral hazard rather than adverse selection. In Albrecht et al. (2016) buyers visit multiple sellers, heterogeneity is two-sided and values are independent.

A Appendix

A.1 Proof of Proposition 3.2

If: We first show that for any vector $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ satisfying conditions (7)-(11), there exists a feasible and incentive compatible mechanism m such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. Consider the following mechanism

$$
\begin{aligned}
\underline{X}_m(L,H) = &\ \underline{\rho}\tfrac{1}{L+\alpha H}, \qquad \underline{T}_m(L,H) = \underline{t}, \quad L \geq 1, \\
\overline{X}_m(L,H) = &\ \overline{\rho}\tfrac{\alpha}{L+\alpha H}, \qquad \overline{T}_m(L,H) = \overline{t}, \quad H \geq 1,
\end{aligned}
$$

for some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$. For the case $\alpha = 0$, let $\overline{X}_m(0,H) = \overline{\rho}\frac{1}{H}$.

This mechanism trivially satisfies $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. We now show that there always exists some tuple $(\alpha, \underline{\rho}, \overline{\rho})$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$. Ex-ante trading probabilities are given by

$$
\begin{aligned}
\underline{x}_m(\underline{\lambda}, \overline{\lambda}) &= \underline{\rho}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\alpha H}, \\
\overline{x}_m(\underline{\lambda}, \overline{\lambda}) &= \overline{\rho}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\alpha}{L+\alpha(H+1)}.
\end{aligned}
$$

Define the function

$$
f(\alpha) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\alpha H}.
$$

Note that $f'(\alpha) < 0$. The function's range is given by $\left[\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}}\right), \frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right)\right]$. To see this, consider first the case $\alpha = 0$:

$$
f(0) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1} = \sum_{L=0}^{+\infty} P_L(\underline{\lambda})\frac{1}{L+1}\underbrace{\sum_{H=0}^{+\infty} P_H(\overline{\lambda})}_{=1} = \frac{1}{\underline{\lambda}}\sum_{L=0}^{+\infty}\frac{\underline{\lambda}^{L+1}}{(L+1)!}e^{-\underline{\lambda}} = \frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right).
$$

Consider next the case $\alpha = 1$:

$$f(1) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+H} = \sum_{N=0}^{+\infty} P_N(\underline{\lambda}+\overline{\lambda})\frac{1}{N+1} = \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right).$$

Next, define the function

$$g(\alpha) = \begin{cases} \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\alpha}{L+\alpha(H+1)} & \text{if } \alpha > 0, \\ \sum_{H=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{H+1} & \text{if } \alpha = 0. \end{cases}$$

Note that $g'(\alpha) > 0$ and that g is continuous at $\alpha = 0$, i.e. $\lim_{\alpha\to 0} g(\alpha) = g(0)$. At $\alpha = 1$, g is equal to f. At $\alpha = 0$, we have

$$g(0) = \sum_{H=0}^{+\infty} P_H(\overline{\lambda})P_0(\underline{\lambda})\frac{1}{H+1} = e^{-\underline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\frac{1}{H+1} = \frac{e^{-\underline{\lambda}}}{\overline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^{H+1}}{(H+1)!}e^{-\overline{\lambda}} = e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right).$$

The range of g is consequently $\left[e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right), \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)\right]$.

With this we can show that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11) we can find some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$ such that $\underline{\rho}f(\alpha) = \underline{x}$ and $\overline{\rho}g(\alpha) = \overline{x}$. Given that $\underline{x}, \overline{x} \geq 0$ and and $\underline{\rho}, \overline{\rho} \in [0,1]$, this can be satisfied if there exists an α such that $f(\alpha) \geq \underline{x}$ and $g(\alpha) \geq \overline{x}$. The first inequality requires that α is not too large, while the second requires that α is not too small. Consider first the case in which $\underline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$. Here $f(\alpha) \geq \underline{x}$ is satisfied for all $\alpha \in [0,1]$. Conditions (7),(8) and (11) together imply $\overline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$, from which it follows that $g(\alpha) \geq \overline{x}$ can be satisfied (e.g. $\alpha = 1$). Consider now the case

$\underline{x} \geq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$ and let $\tilde{\alpha}$ be such that $f(\tilde{\alpha}) = \underline{x}$. We can show

$$
\begin{aligned}
\overline{\lambda} g(\tilde{\alpha}) + \underline{\lambda} f(\tilde{\alpha}) &= \overline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{\tilde{\alpha}}{L+\tilde{\alpha}(H+1)} + \underline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1+\tilde{\alpha} H}, \\
&= \overline{\lambda} \sum_{H=1}^{+\infty} \sum_{L=0}^{+\infty} \frac{\overline{\lambda}^{H-1}}{(H-1)!} \frac{\underline{\lambda}^L}{L!} e^{-\underline{\lambda}-\overline{\lambda}} \frac{\tilde{\alpha}}{L+\tilde{\alpha} H} \frac{H}{H} + \underline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=1}^{+\infty} \frac{\overline{\lambda}^H}{H!} \frac{\underline{\lambda}^{L-1}}{(L-1)!} \frac{1}{L+\tilde{\alpha} H} \frac{L}{L}, \\
&= \sum_{H=1}^{+\infty} \sum_{L=1}^{+\infty} \frac{\overline{\lambda}^H}{H!} \frac{\underline{\lambda}^L}{L!} e^{-\underline{\lambda}-\overline{\lambda}} \left(\underbrace{\frac{\tilde{\alpha} H}{L+\tilde{\alpha} H} + \frac{L}{L+\tilde{\alpha} H}}_{=1} \right) + \sum_{H=1}^{+\infty} \frac{\overline{\lambda}^H}{H!} e^{-\underline{\lambda}-\overline{\lambda}} + \sum_{L=1}^{+\infty} \frac{\underline{\lambda}^L}{L!} e^{-\underline{\lambda}-\overline{\lambda}}, \\
&= \left(1 - e^{-\overline{\lambda}} - e^{-\underline{\lambda}} + e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\underline{\lambda}} - e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\overline{\lambda}} - e^{-\underline{\lambda}-\overline{\lambda}}\right), \\
&= 1 - e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}
$$

With this,

$$
g(\tilde{\alpha}) = \frac{1}{\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}} - \underline{\lambda}, f(\tilde{\alpha})\right) = \frac{1}{\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}} - \underline{\lambda}\, \underline{x}\right) \geq \overline{x},
$$

where the last inequality follows from condition (11). Thus, there exists some $\overline{\rho} \in [0,1]$ such that $\overline{\rho} g(\tilde{\alpha}) = \overline{x}$. Together this implies that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11), there exists some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$.

Finally we need to check feasibility and incentive compatibility of the proposed mechanism. Feasibility follows from

$$
\underline{x}(L,H)L + \overline{x}(L,H)H = \underline{\rho} \frac{1}{L+\alpha H} L + \overline{\rho} \frac{\alpha}{L+\alpha H} H \leq \frac{1}{L+\alpha H} L + \frac{\alpha}{L+\alpha H} H = 1.
$$

Incentive compatibility is trivially satisfied given that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$.

Only if: We now want to show that for any feasible and incentive compatible mechanism m, expected trading probabilities and prices satisfy conditions (7)-(11). Let $\underline{x} = \underline{x}_m(\underline{\lambda}, \overline{\lambda}), \overline{x} = \overline{x}_m(\underline{\lambda}, \overline{\lambda})$ and $\underline{t} = \underline{t}_m(\underline{\lambda}, \overline{\lambda}), \overline{t} = \overline{t}_m(\underline{\lambda}, \overline{\lambda})$. Incentive compatibility of m then trivially implies (7) and (8). Feasibility will imply the remaining conditions. To see this, note first that $\underline{X}_m(L,H)L + \overline{X}_m(L,H)H \leq 1, \forall L, H$ requires $\underline{X}_m(L,H) \leq \frac{1}{L}$, which in turn implies

$$\underline{x}_m(\underline{\lambda},\overline{\lambda}) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\underline{X}_m(L+1,H) \leq \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1} = \frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right).$$

Analogously it can be shown that $\overline{X}_m(L,H) \leq \frac{1}{H}$ implies $\overline{x}_m(\underline{\lambda},\overline{\lambda}) \leq \frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)$. From the perspective of a buyer the probability of trading a low quality good is given by

$$\sum_{L=1}^{+\infty}\sum_{H=0}^{+\infty} \frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L,H)L = \underline{\lambda}\sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} \frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L+1,H) = \underline{\lambda}\,\underline{x}_m(\underline{\lambda},\overline{\lambda}).$$

Similarly, the probability for a buyer to trade a high quality good can be shown to equal $\overline{\lambda}\,\overline{x}_m(\underline{\lambda},\overline{\lambda})$. Feasibility then implies

$$\begin{aligned}
\overline{\lambda}\,\overline{x}_m(\underline{\lambda},\overline{\lambda}) + \underline{\lambda}\,\underline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\left(\underline{X}_m(L,H)L + \overline{X}_m(L,H)H\right), \\
&\leq \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\cdot 1 - P_0(\overline{\lambda})P_0(\underline{\lambda}), \\
&= 1 - e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}$$

□

A.2 Solving P^{aux}: Preliminaries

We will start by deriving some properties on the market utilities that need to be satisfied in equilibrium.

Lemma A.1. *At a competitive search equilibrium, we have*

1.) $\underline{U} > \overline{U}$ *and* $\underline{U} - \overline{U} < \overline{c} - \underline{c}$*;*

2.) $\underline{U}, \overline{U} > 0$*;*

3.) $\underline{U} < \underline{v} - \underline{c}$ *and* $\overline{U} \leq \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$*.*

Proof Let $(\underline{x},\underline{t})$ and $(\overline{x},\overline{t})$ be pairs of expected trading probabilities and transfers associated to (possibly different) mechanisms chosen by low and high type sellers in a given equilibrium. These values must then also be part of a solution of P^{aux}. Market utilities are therefore $\underline{U} = \underline{t} - \underline{x}\,\underline{c}$ and $\overline{U} = \overline{t} - \overline{x}\,\overline{c}$. The following properties must hold:

1a. $\underline{U} > \overline{U}$: the low type incentive constraint (17) requires $\underline{t} - \underline{x}\,\underline{c} \geq \overline{t} - \overline{x}\underline{c}$, which can be rewritten as $\overline{x}(\overline{c} - \underline{c}) \leq \underline{U} - \overline{U}$. Since $\overline{x} \geq 0$, this inequality can only be satisfied if $\underline{U} \geq \overline{U}$. Suppose now that $\underline{U} = \overline{U}$ so that $\overline{x} = 0$. Since, under any solution of P^{aux}, buyers must make weakly positive profits with both types of seller,[45] we must have $\overline{t} = 0$ and hence $\underline{U} = \overline{U} = 0$. A buyer's expected profit with low type sellers when $\underline{U} = \underline{t} - \underline{x}\,\underline{c} = 0$ is $\underline{\lambda}\,\underline{x}(\underline{v} - \underline{c})$. The maximal value of this last expression at an admissible solution of P^{aux} is attained when $\underline{\lambda}\,\underline{x} = (1 - e^{-\underline{\lambda}})$, that is, if the buyer trades with low type sellers whenever possible. However, since $1 - e^{-\underline{\lambda}}$ is strictly increasing in $\underline{\lambda}$, no finite value of $\underline{\lambda}$ can solve P^{aux}, implying that $\underline{U} = \overline{U} = 0$ cannot be admissible equilibrium values.

1b. $\underline{U} - \overline{U} < \overline{c} - \underline{c}$: the high type incentive constraint (8) requires $\overline{t} - \overline{x}\,\overline{c} \geq \underline{t} - \underline{x}\overline{c}$, or $\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}$. Since by condition (10) we have $\underline{x} \leq \frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right) < 1$,[46] the inequality $\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}$ can only be satisfied if $\underline{U} - \overline{U} < \overline{c} - \underline{c}$.

2. $\underline{U}, \overline{U} > 0$: $\underline{U} > 0$ follows directly from $\underline{U} > \overline{U}$ and $\overline{U} \geq 0$. It thus remains to show that $\overline{U} > 0$. Towards a contradiction, suppose $\overline{U} = 0$. In this case, by a symmetric argument to the one in 1a. above, the expected profit a buyer makes trading with high type sellers is $\overline{\lambda}\,\overline{x}(\overline{v} - \overline{c})$. Since $\underline{U} > \overline{U}$, there always exists a strictly positive value of $\overline{x}$ that satisfies the low type incentive constraint (7) and the overall feasibility constraint (11). Given that $\overline{\lambda}$ can be set equal to zero, a strictly positive value of $\overline{x}$ is also weakly optimal. Thus, w.l.o.g. suppose $\overline{x} > 0$ and consider an increase of $\overline{\lambda}$ together with a decrease of $\overline{x}$ so as to keep $\overline{\lambda}\,\overline{x}$ unchanged. Adjusting $\overline{t}$ in order to keep the utility of high type sellers constant, while keeping the remaining contracting parameters unchanged, we obtain a tuple $(\underline{\lambda}, \overline{\lambda}', \underline{x}, \overline{x}', \underline{t}, \overline{t})$ that is incentive compatible and satisfies the overall feasibility constraint (11) with strict inequality, i.e. $\underline{\lambda}\,\underline{x} + \overline{\lambda}'\overline{x}' < 1 - e^{-\underline{\lambda} - \overline{\lambda}'}$. The buyer can then increase his payoff by deviating to $\left(\underline{\lambda}, \overline{\lambda}' + \varepsilon, \underline{x}, \overline{x}', \underline{t}, \overline{t}'\right)$, with $\varepsilon > 0$, while still satisfying all constraints of P^{aux}, thus a contradiction.

3a. $\underline{U} < \underline{v} - \underline{c}$: Suppose not, $\underline{U} \geq \underline{v} - \underline{c}$. Since, as shown in 1b. above, $\underline{x} < 1$ whenever $\underline{\lambda} > 0$, this implies that $\underline{x}(\underline{v} - \underline{c}) - \overline{U} < 0$, that is, a buyer's payoff with each low type seller is strictly negative. As a consequence, at any solution of P^{aux} we have $\underline{\lambda} = 0$. This in turn implies that the low types' market utility $\underline{U}$ must equal zero and therefore

[45] If buyers make losses with one type of seller, they can always set the respective ratio, $\underline{\lambda}$ or $\overline{\lambda}$, equal to zero.

[46] $\frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right) < 1$ is a general property for all $\underline{\lambda} \in (0, +\infty)$.

$\underline{U} < \underline{v} - \underline{c}$. A contradiction.

3b. $\overline{U} \leq \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$: Suppose not, $\overline{U} > \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. As argued above, the low type incentive constraint (7) can be written as $\overline{x} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. This implies that the payoff of a buyer with each high type seller is negative:

$$\overline{x}(\overline{v}-\overline{c}) - \overline{U} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) - \overline{U} = \frac{(\overline{v}-\overline{c})\underline{U} - (\overline{v}-\underline{c})\overline{U}}{\overline{c}-\underline{c}} < 0.$$

All solutions of P^{aux} must therefore satisfy $\overline{\lambda} = 0$, which in turn implies $\overline{U} = 0$ and therefore $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. A contradiction.

□

Next, we can show that w.l.o.g. the participation constraints of the two types of sellers in P^{aux} can be set holding as equalities. Consider first the possibility that at a solution of P^{aux} the buyer only wants to attract high-type sellers, i.e. $\underline{\lambda} = 0$. Since $\underline{U} - \overline{U} < \overline{c} - \underline{c}$, there exists a pair $\underline{t}, \underline{x}$ so as to satisfy $\underline{t} - \underline{x}\underline{c} = \underline{U}$ and $\underline{t} - \underline{x}\overline{c} \leq \overline{U}$. In particular, any combination of $\underline{x}$ and $\underline{t}$ that satisfies $\underline{x} \geq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ and $\underline{t} = \underline{U} + \underline{x}\underline{c}$ makes low type sellers indifferent and satisfies the incentive constraint of high type sellers. The actual choice of $\underline{x}, \underline{t}$ does not affect the buyer's payoff, as $\underline{x}$ and $\underline{t}$ are multiplied by $\underline{\lambda} = 0$ both in the objective and in the remaining constraints. A symmetric argument can be made for the case of $\overline{\lambda} = 0$.

Solving the participation constraints for $\underline{t}, \overline{t}$ and substituting into the objective function and the remaining constraints, the buyer's optimization problem can be rewritten in the following simpler form

$$\max_{\underline{x},\overline{x},\underline{\lambda},\overline{\lambda}} \quad \overline{\lambda}\left[\overline{x}(\overline{v}-\overline{c}) - \overline{U}\right] + \underline{\lambda}\left[\underline{x}(\underline{v}-\underline{c}) - \underline{U}\right], \qquad (P^{aux'})$$

subject to

$$\overline{x}(\overline{c}-\underline{c}) \leq \underline{U} - \overline{U}, \qquad (17)$$

$$\underline{x}(\overline{c}-\underline{c}) \geq \underline{U} - \overline{U}, \qquad (18)$$

$$\underline{\lambda}\,\underline{x} \leq 1 - e^{-\underline{\lambda}}, \qquad (19)$$

$$\overline{\lambda}\,\overline{x} + \underline{\lambda}\,\underline{x} \leq 1 - e^{-\overline{\lambda}-\underline{\lambda}}, \qquad (20)$$

$$\overline{\lambda}, \underline{\lambda} \geq 0. \qquad (21)$$

All properties of P^{aux} will be proven by considering the simplified problem $P^{aux'}$.

A.3 Proof of Lemma 3.3

Towards a contradiction, suppose there is a solution $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ of $P^{aux'}$ with $\underline{x}\,\underline{\lambda} < 1 - e^{-\underline{\lambda}}$. Note that the strict inequality requires $\underline{\lambda} > 0$. Consider then an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' < \underline{\lambda}$, $\overline{\lambda}' = \overline{\lambda} + (\underline{\lambda} - \underline{\lambda}')$, $\underline{x}' = \frac{\underline{\lambda}}{\underline{\lambda}'}\underline{x}$ and $\overline{x}' = \frac{\overline{\lambda}}{\overline{\lambda}'}\overline{x}$. If $\underline{\lambda}'$ is sufficiently close to $\underline{\lambda}$, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies constraint (19). Since $\underline{x}'\underline{\lambda}' = \underline{x}\,\underline{\lambda}$, $\overline{x}'\overline{\lambda}' = \overline{x}\,\overline{\lambda}$ and $\underline{\lambda}' + \overline{\lambda}' = \underline{\lambda} + \overline{\lambda}$, it also satisfies the overall feasibility constraint (20). Lastly, since $\underline{x}' > \underline{x}$ and $\overline{x}' < \overline{x}$, the mechanism associated to this alternative tuple always satisfies the incentive compatibility constraints (17) and (18). Consider now the buyer's payoff associated to $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$:

$$\begin{aligned}
& \underline{\lambda}'\underline{x}'(\underline{v} - \underline{c}) + \overline{\lambda}'\overline{x}'(\overline{v} - \overline{c}) - \underline{\lambda}'\underline{U} - \overline{\lambda}'\overline{U}, \\
= \quad & \underline{\lambda}\,\underline{x}(\underline{v} - \underline{c}) + \overline{\lambda}\,\overline{x}(\overline{v} - \overline{c}) - \underline{\lambda}'\underline{U} - (\overline{\lambda} + (\underline{\lambda} - \underline{\lambda}'))\overline{U}, \\
= \quad & \underline{\lambda}\,\underline{x}(\underline{v} - \underline{c}) + \overline{\lambda}\,\overline{x}(\overline{v} - \overline{c}) - \underline{\lambda}\,\underline{U} - \overline{\lambda}\,\overline{U} + (\underline{\lambda} - \underline{\lambda}')(\underline{U} - \overline{U}).
\end{aligned}$$

Since the last term of the last line is strictly positive, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ yields a strictly higher payoff than $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$, a contradiction. □

A.4 Proof of Lemma 3.4

Given market utilities, a buyer's profit per high type seller he attracts is $\overline{x}(\overline{v} - \overline{c}) - \overline{U}$. The maximal incentive feasible trading probability for high type sellers is $\overline{x} = \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}$ (see constraint (17) in $P^{aux'}$). Substituting this value of the trading probability into the profits a buyer obtains with each high type seller, we find that these profits are non-negative if $\overline{U} \leq \frac{\overline{v} - \overline{c}}{\overline{v} - \underline{c}}\underline{U}$.

i) We start by considering the case, where this inequality holds as a strict inequality so that buyers can make strictly positive profits with high type seller. We can then show that at all solutions of $P^{aux'}$ the overall feasibility condition (20) is satisfied as equality. Let $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ be a tuple that solves $P^{aux'}$. If $\overline{\lambda} = 0$, constraint (20) is satisfied as equality by Lemma 3.3. Consider now the case $\overline{\lambda} > 0$. Towards a contradiction, suppose that (20) is satisfied as a strict inequality at a solution of $P^{aux'}$. Under this assumption, if the buyer's profit with each high type seller, $\overline{x}(\overline{v} - \overline{c}) - \overline{U}$, is strictly positive, the buyer can strictly increase his payoff by increasing $\overline{\lambda}$. For this not to be a profitable deviation, we must therefore have $\overline{x}(\overline{v}-\overline{c}) \leq \overline{U}$. Noting that the stated property $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$ is equivalent to $\frac{\overline{U}}{\overline{v}-\overline{c}} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, the inequality $\overline{x}(\overline{v}-\overline{c}) \leq \overline{U}$ implies $\overline{x} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. That is, the low type incentive constraint (17) is satisfied as a strict inequality. By slackness of the constraints (17) and (20), an increase in $\overline{x}$ is then both feasible and incentive compatible. Since increasing $\overline{x}$ strictly increases the buyer's payoff with each high type seller and since $\overline{\lambda} > 0$, the buyer has a profitable deviation, thus a contradiction.

Having shown that, when $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, both feasibility constraints (19) and (20) are satisfied as equality at a solution of $P^{aux'}$, the values of $\underline{\lambda}, \overline{\lambda}$ belonging to a solution of $P^{aux'}$ maximize

$$\hat{\pi}(\underline{\lambda}, \overline{\lambda}) := \left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c}) - \underline{\lambda}\,\underline{U} - \overline{\lambda}\,\overline{U}. \tag{22}$$

subject to incentive compatibility constraints (17) and (18), which can, respectively, be rewritten as

$$e^{-\underline{\lambda}}(1-e^{-\overline{\lambda}}) \leq \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}, \tag{23}$$

$$(1-e^{-\underline{\lambda}}) \geq \underline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{24}$$

a) Consider first the case $\underline{v}-\underline{c} > \overline{v}-\overline{c}$. Under this condition, $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ is strictly concave. To see this, note that $\frac{\partial^2 \hat{\pi}}{\partial \overline{\lambda}^2} = -e^{-\underline{\lambda}}e^{-\overline{\lambda}}(\overline{v}-\overline{c}) < 0$ and that the determinant of the Hessian is given by $e^{-2\underline{\lambda}}e^{-\overline{\lambda}}(\overline{v}-\overline{c})[(\underline{v}-\underline{c})-(\overline{v}-\overline{c})] > 0$. The Hessian is thus negative definite.
   We can show that, as a consequence, there cannot exist two solutions $(\underline{x}_1, \overline{x}_1, \underline{\lambda}_1, \overline{\lambda}_1)$ and $(\underline{x}_2, \overline{x}_2, \underline{\lambda}_2, \overline{\lambda}_2)$. Strict concavity of $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ would in fact imply that a strictly larger payoff is attained at $\underline{\lambda} = \alpha\underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2, \overline{\lambda} = \alpha\overline{\lambda}_1 + (1-\alpha)\overline{\lambda}_2$, $\alpha \in (0,1)$. The pair $(\underline{\lambda}, \overline{\lambda})$ satisfies the incentive constraints since $\frac{1}{\underline{\lambda}}(1-e^{-\underline{\lambda}}) \geq \text{Min}\left\{\frac{1}{\underline{\lambda}_1}(1-e^{-\underline{\lambda}_1}), \frac{1}{\underline{\lambda}_2}(1-e^{-\underline{\lambda}_2})\right\}$ and $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}(1-e^{-\overline{\lambda}}) \leq \text{Max}\left\{e^{-\underline{\lambda}_1}\frac{1}{\overline{\lambda}_1}(1-e^{-\overline{\lambda}_1}), e^{-\underline{\lambda}_2}\frac{1}{\overline{\lambda}_2}(1-e^{-\overline{\lambda}_2})\right\}$.[47] Hence $\left(\underline{\lambda}, \overline{\lambda}, \underline{x}, \overline{x}\right)$ is an admissible solution and yields a higher payoff, a contradiction.

b) Consider next the case $\underline{v}-\underline{c} \leq \overline{v}-\overline{c}$. We will first establish that the incentive constraint of the low type sellers must be binding at a solution of $P^{aux'}$

[47]It can be verified that $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}(1-e^{-\overline{\lambda}})$ is convex in $\underline{\lambda}$ and $\overline{\lambda}$.

Lemma A.2. *If* $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$ *and* $\underline{v}-\underline{c} \leq \overline{v}-\overline{c}$, *at any solution of* $P^{aux'}$ *the low type incentive compatibility constraint (17) is satisfied with equality.*

Proof We argue by contradiction: assume $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ is a solution of $P^{aux'}$ (with $\overline{\lambda} > 0$) and (17) is satisfied as inequality: $\overline{x} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Suppose first that $\underline{\lambda} > 0$ and consider an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' = \underline{\lambda} - \Delta, \Delta > 0$, $\overline{\lambda}' = \overline{\lambda} + \Delta$, $\underline{x}' = \frac{1}{\underline{\lambda}'}\left(1 - e^{-\underline{\lambda}'}\right)$ and $\overline{x}' = e^{-\underline{\lambda}'}\frac{1}{\overline{\lambda}'}(1 - e^{-\overline{\lambda}'})$. In the alternative tuple some low types are replaced with high types, while the ratio between buyers and all types of sellers is kept unchanged, and the feasibility constraints (19) and (20) still hold as equality. As a consequence we have $\underline{x}' > \underline{x}$ and $\overline{x}' > \overline{x}$, implying that $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies all constraints of problem $P^{aux'}$ as long as Δ is sufficiently small. The difference between the buyer's payoff associated to $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ and that associated to $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ is then:

$$\hat{\pi}(\underline{\lambda}', \overline{\lambda}') - \hat{\pi}(\underline{\lambda}, \overline{\lambda}) = \left(e^{-\underline{\lambda}+\Delta} - e^{-\underline{\lambda}}\right)\left[(\overline{v}-\overline{c}) - (\underline{v}-\underline{c})\right] + \Delta(\underline{U} - \overline{U}) > 0.$$

The first term is the difference in the probability of meeting no low type seller (in which case a high quality good is traded), multiplied by the difference between the gains from trade of the high and low quality good, while the second term is the difference in rent paid to sellers: since the alternative mechanism on average replaces Δ low type sellers with high type sellers, the reduction in rent is $\Delta(\underline{U} - \overline{U})$. Under the assumption $\underline{v}-\underline{c} \leq \overline{v}-\overline{c}$, the sum of the two terms is strictly positive. Hence, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ yields a strictly larger payoff than $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$, a contradiction.

For the case $\underline{\lambda} = 0$, notice that $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ as well as the correspondence determining the set of feasible values of $\overline{\lambda}$, defined by the constraint $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, are continuous in $\underline{\lambda}$ at $\underline{\lambda} = 0$. Consider the constrained optimization problem where we require $\underline{\lambda} \geq \varepsilon > 0$. By the previous argument, incentive constraint (17) holds as equality at a solution of this constrained problem for all $\varepsilon > 0$, and by the continuity property the solution of the constrained problem converges to the solution of the original problem where we only require $\underline{\lambda} \geq 0$. Since (17) holds as equality along all points in the sequence, it does so in the limit. □

Having established that the low type incentive constraint (17) holds as equality at all solutions with $\overline{\lambda} > 0$, we can solve this constraint for $\underline{\lambda}$, yielding:

$$\underline{\lambda} = \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right).$$

The condition $\underline{\lambda} \geq 0$ requires that $\overline{\lambda} \leq L$, where L is defined by $\frac{1}{L}(1-e^{-L}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. For $\overline{\lambda} \in (0, L]$, the buyer's payoff can then be written as a function of $\overline{\lambda}$ only:

$$\tilde{\pi}(\overline{\lambda}) = \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) + \left(1 - \frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}\right)(\underline{v}-\underline{c}) - \overline{\lambda}\,\overline{U} - \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)\underline{U}. \tag{25}$$

Since $\underline{x} > \overline{x} =$ for $\underline{\lambda}, \overline{\lambda} > 0$ (by Lemma 3.3), the fact that the low type incentive constraint (17) holds as equality immediately implies that the high type incentive constraint (18) is slack. At a solution of $P^{aux'}$ with $\overline{\lambda} > 0$, $\overline{\lambda}$ is thus the unconstrained maximizer of $\tilde{\pi}$ on $(0, L]$. We show next that the function $\tilde{\pi}$ is strictly concave in $\overline{\lambda}$. Note that:

$$\frac{\partial^2\tilde{\pi}}{\partial\overline{\lambda}^2} = e^{\overline{\lambda}}\frac{2(e^{\overline{\lambda}}-1) - \overline{\lambda}(e^{\overline{\lambda}}+1)}{(e^{\overline{\lambda}}-1)^3}(\underline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} + \frac{\overline{\lambda}^2 e^{\overline{\lambda}} - (e^{\overline{\lambda}}-1)^2}{\overline{\lambda}^2(e^{\overline{\lambda}}-1)^2}\underline{U}. \tag{26}$$

The numerator of the first term of the derivative is equal to zero at $\overline{\lambda} = 0$ and strictly decreasing for all $\overline{\lambda} > 0$:

$$\frac{\partial\left(2(e^{\overline{\lambda}}-1) - \overline{\lambda}(e^{\overline{\lambda}}+1)\right)}{\partial\overline{\lambda}} = -\overline{\lambda}e^{\overline{\lambda}}\underbrace{\left(1 - \frac{1}{\overline{\lambda}}(1-e^{-\overline{\lambda}})\right)}_{>0} < 0,$$

so that this first term is strictly negative. As we show next, the numerator of the second term is also strictly negative: $\overline{\lambda}^2 e^{\overline{\lambda}} < (e^{\overline{\lambda}}-1)^2$. To see this, notice first that the inequality can be rewritten as $1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\frac{\overline{\lambda}}{2}} < 0$. The term $1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\frac{\overline{\lambda}}{2}}$ is equal to zero at $\overline{\lambda} = 0$ and is strictly decreasing in $\overline{\lambda}$ for all $\overline{\lambda} > 0$:

$$\frac{\partial\left(1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\frac{\overline{\lambda}}{2}}\right)}{\partial\overline{\lambda}} = -e^{\overline{\lambda}}\left[1 - e^{-\frac{\overline{\lambda}}{2}} - \frac{\overline{\lambda}}{2}e^{-\frac{\overline{\lambda}}{2}}\right] < 0,$$

where the term in the square bracket is the probability of at least two arrivals when the queue length is $\frac{\overline{\lambda}}{2}$ and therefore is strictly positive. This establishes $\frac{\partial^2\tilde{\pi}}{\partial\overline{\lambda}^2} < 0$, that is $\tilde{\pi}(\overline{\lambda})$ is strictly concave on the domain $(0, L]$.

Given the strict concavity of $\tilde{\pi}(\overline{\lambda})$ on $(0, L]$, $P^{aux'}$ can at most have two solutions, one at $\overline{\lambda} > 0$ and possibly one at $\overline{\lambda} = 0$. In what follows, we show there cannot be a solution with $\overline{\lambda} = 0$, which establishes the claim that the solution of $P^{aux'}$ is unique. To this end, we need to characterize the properties of possible solutions with $\overline{\lambda} = 0$. Recall that the above expression of $\tilde{\pi}(\overline{\lambda})$ was only valid for $\overline{\lambda} > 0$, hence when $\overline{\lambda} = 0$ we need to consider $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$. Ignoring for a moment incentive constraints, the value of $\underline{\lambda}$ that maximizes $\hat{\pi}(\underline{\lambda}, 0) = (1 - e^{-\underline{\lambda}})(\underline{v} - \underline{c}) - \underline{\lambda}\,\underline{U}$, is $\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. Given $\overline{\lambda} = 0$, the high type incentive constraint (17) can always be satisfied by picking some value of $\overline{x}$ weakly smaller than $\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. On the other hand, the high type incentive constraint (18) is satisfied at $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$ if and only if

$$\frac{1}{\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)}\left(1 - \frac{\underline{U}}{\underline{v} - \underline{c}}\right) \geq \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}. \tag{27}$$

- Suppose first that this inequality is not satisfied, that is the value of $\underline{\lambda}$ maximizing $\hat{\pi}(\underline{\lambda}, 0)$ is too large. The optimal value of $\underline{\lambda}$ is then given by L, defined earlier by the implicit condition $\frac{1}{L}(1 - e^{-L}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. The buyer's payoff associated to this value of $\underline{\lambda}$ is $\hat{\pi}(L, 0) = L\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\underline{v} - \underline{c}) - \underline{U}\right)$. Since $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$ and $\underline{U} > \overline{U}$, $\hat{\pi}(L, 0)$ is strictly smaller than the corresponding expression when all low types are swapped with high types, $\hat{\pi}(0, L) = L\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v} - \overline{c}) - \overline{U}\right)$. Since the pair $(0, L)$ is an admissible solution for $(\underline{\lambda}, \overline{\lambda})$, we cannot have $\overline{\lambda} = 0$ at a solution of $P^{aux'}$, hence a contradiction.
- Suppose next that (27) is satisfied and so that $\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$ is an admissible value for $\underline{\lambda}$. Towards a contradiction, suppose that we have $\underline{\lambda} = \ln(\frac{\underline{v}-\underline{c}}{\underline{U}})$ and $\overline{\lambda} = 0$ at a solution of $P^{aux'}$. We need to distinguish two cases:
  * Suppose first that $\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Consider the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' = \underline{\lambda}$ and $\underline{x}' = \frac{1}{\underline{\lambda}}(1 - e^{-\underline{\lambda}})$, so that the payoff with low type sellers remains unchanged, and let $\overline{x}' = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Given these restrictions, the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies the incentive compatibility constraints and, if $\overline{\lambda}' > 0$, yields a strictly

positive payoff with high type sellers:

$$\overline{\lambda}' \left[\overline{x}'(\overline{v}-\overline{c})-\overline{U}\right] = \overline{\lambda}\left[\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c})-\overline{U}\right].$$

A strictly positive value of $\overline{\lambda}'$ is feasible (that is, consistent with (20), using the values specified above for $\underline{\lambda}'$, $\underline{x}', \overline{x}'$) if

$$\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} \leq \frac{\underline{U}}{\underline{v}-\underline{c}}\frac{1}{\overline{\lambda}'}\left(1-e^{-\overline{\lambda}'}\right),$$

for some $\overline{\lambda}' > 0$. In the case under consideration ($\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$), this is indeed the case, implying that there is no solution of $P^{aux'}$ with $\overline{\lambda} = 0$.

∗ Consider next the case $\frac{\underline{U}}{\underline{v}-\underline{c}} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Let $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ be specified again with $\underline{\lambda}' = \underline{\lambda}$, $\underline{x}' = \frac{1}{\underline{\lambda}}(1-e^{-\underline{\lambda}})$, while the value of $\overline{x}'$ is now set so that (20) is satisfied with equality:

$$\overline{x}' = e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}'}\left(1-e^{-\overline{\lambda}'}\right) = \frac{\underline{U}}{\underline{v}-\underline{c}}\frac{1}{\overline{\lambda}'}\left(1-e^{-\overline{\lambda}'}\right) < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}.$$

The above inequality implies that the low type incentive constraint (17) is satisfied for all $\overline{\lambda}' > 0$.[48] The difference in payoff between the two mechanisms is given by:

$$\begin{aligned} &\hat{\pi}\left(\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), 0\right) - \hat{\pi}\left(\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}'\right), \\ = \ & e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}'}\right)(\overline{v}-\overline{c}) - \overline{\lambda}'\overline{U} = \overline{\lambda}'\left[\frac{1}{\overline{\lambda}'}\left(1-e^{-\overline{\lambda}'}\right)\frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U} - \overline{U}\right]. \end{aligned}$$

Since $\underline{U} > \overline{U}$ and $\overline{v}-\overline{c} \geq \underline{v}-\underline{c}$ in the case under consideration (b), the above expression is strictly positive for $\overline{\lambda}'$ small enough.[49]

We have thus shown there can be no solution of $P^{aux'}$ with $\overline{\lambda} = 0$ and therefore that the solution of $P^{aux'}$ is unique.

□

[48] The other incentive compatibility constraint (18) is also satisfied because the low type sellers' trading probability remains unchanged.

[49] Recall that $\lim_{x\to 0}\frac{1}{x}(1-e^{-x}) = 1$.

ii) We next consider the case where $\overline{U} = \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$ so that buyers make at most zero profits with high type sellers. In this case we have $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c}) - \overline{U}] = 0$ for all solutions of $P^{aux'}$ and $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ for all solutions of $P^{aux'}$ with $\overline{\lambda} > 0$. We can then show that $\underline{\lambda}$ is the same at all solutions of $P^{aux'}$. Suppose not. By Lemma 3.3, the buyer's payoff from low type sellers is given by $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c}) - \underline{\lambda}\,\underline{U}$. This term is strictly concave in $\underline{\lambda}$ and attains its maximum at $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. This implies that any solution of $P^{aux'}$ must satisfy $\underline{\lambda} \leq \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$: if in fact the buyer proposes a mechanism with $\underline{\lambda} > \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$, decreasing $\underline{\lambda}$ and adjusting $\underline{x}$ so that (19) still holds as equality strictly increases the buyer's payoff and weakly relaxes all constraints. Now suppose there exists two values $\underline{\lambda}_1$ and $\underline{\lambda}_2$ with $\underline{\lambda}_1 \neq \underline{\lambda}_2$ that belong to some solution of $P^{aux'}$. W.l.o.g. assume $\underline{\lambda}_1 < \underline{\lambda}_2$, which implies $(1-e^{-\underline{\lambda}_1})(\underline{v}-\underline{c}) - \underline{\lambda}_1\underline{U} < (1-e^{-\underline{\lambda}_2})(\underline{v}-\underline{c}) - \underline{\lambda}_2\underline{U}$. Since both solutions must yield the same payoff for the buyer, we must then have $\overline{\lambda}_1[\overline{x}_1(\overline{v}-\overline{c}) - \overline{U}] > \overline{\lambda}_2[\overline{x}_2(\overline{v}-\overline{c}) - \overline{U}]$. However, we know that $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c}) - \overline{U}] = 0$ for all solutions of $P^{aux'}$ and therefore have a contradiction.

Finally, the property that $\underline{\lambda}$ is the same for all solutions of $P^{aux'}$, together with the result in Lemma 3.3, immediately implies that $\underline{x}$ must have the same value for all solutions of $P^{aux'}$. Through the low type participation constraint (13), this in turn implies that also the transfer $\underline{t}$ is the same. □

A.5 Proof of Proposition 3.5

We start with the assumption that market utilities are such that buyers can make positive profits with high type sellers, i.e. $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$. The proof of Lemma 3.4 showed that in this case the overall feasibility constraint is binding, so we can use (14) to substitute for $\underline{x}$ and $\overline{x}$ and write $P^{aux'}$ solely in terms of the variables $\underline{\lambda}, \overline{\lambda}$, as already done in the proof of Lemma 3.4. At a solution of $P^{aux'}$, the values of $\underline{\lambda}, \overline{\lambda}$ must then maximize $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$, specified in (22), subject to the incentive constraints, as rewritten in (23) and (24).

a) Suppose first the incentive constraints are not binding. The first order conditions for an interior solution with respect to $\underline{\lambda}$ and $\overline{\lambda}$ are:

$$e^{-\underline{\lambda}}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c}) - \underline{U} = 0, \quad (28)$$

$$e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c}) - \overline{U} = 0. \quad (29)$$

The solution of this system of equations is equal to $\underline{\lambda}^p, \overline{\lambda}^p$ if and only if market utilities

$\underline{U}$ and $\overline{U}$ are given by

$$\underline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}) + e^{-\underline{\lambda}^p}[(\underline{v}-\underline{c})-(\overline{v}-\overline{c})], \tag{30}$$
$$\overline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}); \tag{31}$$

Consider then the buyers' payoff with high type sellers

$$\overline{x}(\overline{v}-\overline{c}) - \overline{U} = e^{-\overline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}-\overline{\lambda}^p e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c}).$$

We have $1-e^{-\overline{\lambda}^p} > \overline{\lambda}^p e^{-\overline{\lambda}^p}$ because $1-e^{-\overline{\lambda}^p}$ equals the probability of at least one arrival when the queue length is $\overline{\lambda}^p$, and this strictly exceeds the probability of exactly one arrival, $\overline{\lambda}^p e^{-\overline{\lambda}^p}$. The buyers' payoff with high type sellers at the equilibrium mechanism is therefore strictly positive, which implies that market utilities as given in (30) and (31) satisfy the assumed condition $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$.

Finally, we need to check under which conditions incentive compatibility constraints (23) and (24) hold at $\underline{\lambda}^p, \overline{\lambda}^p$. Substituting the values of $\underline{U}$ and $\overline{U}$ as in (30) and (31) into these constraints, we obtain the following inequalities:

$$\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \le 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}, \tag{32}$$
$$\frac{1-e^{-\underline{\lambda}^p}}{\underline{\lambda}^p e^{-\underline{\lambda}^p}} \ge 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}. \tag{33}$$

Since $1-e^{-\underline{\lambda}^p} > \underline{\lambda}^p e^{-\underline{\lambda}^p}$, as we just argued, inequality (33) is always satisfied. The described equilibrium thus exists if and only if (32) is satisfied.

b) Suppose next at least one of the incentive constraints (23), (24) is binding. Letting γ_l and γ_h denote the respective Lagrange multipliers of these constraints, the population parameters $\underline{\lambda}^p, \overline{\lambda}^p$ are optimal if they solve the first-order conditions:

$$e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c}) - \underline{U} + \gamma_l e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p}) - \gamma_h\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}^p}\right) = 0,$$
$$e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}) - \overline{U} + \gamma_l\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}^p-\overline{\lambda}^p}\right) = 0.$$

Consider first the possibility that the high type incentive constraint (24) is binding, i.e. $\gamma_h > 0, \gamma_l = 0$. In this case $\overline{U}$ has the same value as in case a), given by (31), and $\frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Since $e^{-\underline{\lambda}^p} < \frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, the term multiplying γ_h is positive. For γ_h to be strictly positive the following condition must then hold:

$$e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c}) - \underline{U} > 0.$$

With $\overline{U}$ determined by (31) and $\underline{U}$ such that $\frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, this inequality can be rewritten as

$$\frac{1-e^{-\underline{\lambda}^p}}{\underline{\lambda}^p e^{-\underline{\lambda}^p}} < 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}.$$

This inequality is the complement of (33). Following the argument above, it is always violated and so we must have $\gamma_h = 0$.

Consider next the case where the low type incentive constraint (23) binds: $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. As shown in the proof of part i.b) of Lemma 3.4, solving this equation for $\underline{\lambda}$ and substituting the result into the expression of the buyer's payoff in (22) yields the function $\tilde{\pi}(\overline{\lambda})$, specified in (25). The solution of the buyer's auxiliary problem is then determined by the first-order condition $\tilde{\pi}'(\overline{\lambda}) = 0$. Substituting the population parameter $\overline{\lambda}^p$ into this equality we get the following condition:

$$(\overline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{\overline{\lambda}^p}\frac{e^{\overline{\lambda}^p}-\overline{\lambda}^p-1}{(e^{\overline{\lambda}^p}-1)^2}(\underline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} + \frac{e^{\overline{\lambda}^p}-\overline{\lambda}^p e^{\overline{\lambda}^p}-1}{\overline{\lambda}^p(e^{\overline{\lambda}^p}-1)}\underline{U} = 0. \tag{34}$$

Solving this equation, together with $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, for $\underline{U}$ and $\overline{U}$ yields

$$\underline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)\frac{1-e^{-\overline{\lambda}^p}}{1-\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p})}(\overline{v}-\underline{v}), \tag{35}$$

$$\overline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)\left[\frac{1-e^{-\overline{\lambda}^p}}{1-\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p})}(\overline{v}-\underline{v}) - (\overline{c}-\underline{c})\right]. \tag{36}$$

It is immediate to verify that these values satisfy the condition $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ if and only if $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. This condition implies that at the maximal incentive feasible trading probability for high type sellers buyers make strictly positive profits with high type sellers. Since the low type incentive constraint is binding, this is indeed the case. $\square$

A.6 Proof of Proposition 3.6

We start by assuming that buyers can make at most zero profits with high type sellers: $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. The proof of Lemma 3.4 (ii) showed that in this case we have $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, while $\underline{x}$ is determined by (14). Problem $P^{aux'}$ then corresponds to maximizing $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c}) - \underline{\lambda}\,\underline{U}$ over $\underline{\lambda}$ subject to the incentive constraint of high type sellers, as rewritten in (24). This constraint is not binding, given the above value of $\bar{x}$ and the property $\underline{x} > \overline{x}$ we established. With the high type incentive constraint (18) not binding, the optimal value of $\underline{\lambda}$ is uniquely characterized by the first-order condition

$$e^{-\underline{\lambda}}(\underline{v}-\underline{c}) - \underline{U} = 0.$$

Since all buyers find it optimal to attract the same queue length of low type seller, consistency with the population parameters requires that this solution is given by $\underline{\lambda}^p$, which requires

$$\underline{U} = e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}). \tag{37}$$

The value of $\overline{U}$ in the candidate equilibrium is then pinned down by the assumed condition $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ and is so given by

$$\overline{U} = e^{-\underline{\lambda}^p}\frac{(\underline{v}-\underline{c})(\overline{v}-\overline{c})}{\overline{v}-\underline{c}}. \tag{38}$$

With $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, $P^{aux'}$ is solved by all values of $\overline{\lambda}$ that satisfy the overall feasibility constraint (20). Substituting $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ evaluated at the market utilities in (37) and (38), and $\underline{x}$ as determined by (14) and $\underline{\lambda}^p$, this constraint reduces to

$$\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}. \tag{39}$$

An equilibrium is then given by any distribution of ratios between high type sellers and buyers G with support Λ such that $\int_\Lambda \overline{\lambda} dG(\overline{\lambda}) = \overline{\lambda}^p$ and feasibility constraint (39) is satisfied for all values of $\overline{\lambda}$ in Λ. Since $\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)$ is decreasing in $\overline{\lambda}$ and since $\max\Lambda \geq \overline{\lambda}^p$, such distribution exists if and only if (39) is satisfied at $\overline{\lambda} = \overline{\lambda}^p$.

□

A.7 Proof of Proposition 4.1

The statement that the competitive search equilibrium allocation maximizes total surplus if $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ and $\underline{v}-\underline{c} \geq \overline{v}-\overline{c}$ follows from the argument in the text, together with the property that under urn-ball matching having buyers and sellers trade in a single submarket maximizes the number of meetings (formally shown in the Online Appendix). We now demonstrate that both conditions are necessary for total surplus to be maximized and that, if they are not satisfied, the allocation is constrained inefficient whenever μ is sufficiently large.

- Consider first the case $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. Proposition 3.6 shows that under this condition at a competitive search equilibrium the overall feasibility constraint (11) is slack. Consider an increase in the trading probability of the high type seller $\Delta\overline{x}$, small enough so that (11) is not violated. Modify then the expected transfer to the high type seller so that his utility is kept constant

  $$\Delta\overline{t} = \Delta\overline{x}\,\overline{c}.$$

  The trading probability of the low type seller is kept unchanged and the expected transfer to the low type seller is adjusted to ensure that his incentive compatibility constraint is satisfied

  $$\Delta\underline{t} - \underbrace{\Delta\underline{x}\,\underline{c}}_{=0} = \Delta\overline{t} - \Delta\overline{x}\underline{c} \quad \Leftrightarrow \quad \Delta\underline{t} = (\overline{c}-\underline{c})\,\Delta\overline{x}.$$

  These changes make the high type sellers indifferent, strictly improve the low type sellers' (since $\Delta\underline{t} > 0$), and increase the total surplus that is generated. They also make buyers weakly better off and thus constitute a Pareto improvement if

  $$\overline{\lambda}^p\left[\Delta\overline{x}\,\overline{v} - \Delta\overline{t}\right] + \underline{\lambda}^p\left[\Delta\underline{x}\,\underline{v} - \Delta\underline{t}\right] \geq 0, \quad \Leftrightarrow \quad \frac{\underline{\lambda}^p}{1-\mu}\left[\mu(\overline{v}-\underline{c}) - (\overline{c}-\underline{c})\right]\Delta\overline{x} \geq 0,$$

  which is satisfied whenever $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{c}} \in (0,1)$. □

- Next, consider the case $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ together with $\underline{v}-\underline{c} < \overline{v}-\overline{c}$. Under this specification the overall feasibility constraint (11) is binding (see Propositions 3.5 and 3.6). Consider an increase in the trading probability of the high type seller by $\Delta\overline{x}$, while adjusting the trading probability of the low type seller so that the feasibility constraint is still satisfied as equality:

$$\overline{\lambda}^p \Delta \overline{x} + \underline{\lambda}^p \Delta, \underline{x} = 0 \quad \Leftrightarrow \quad \Delta \underline{x} = -\frac{\mu}{1-\mu} \Delta \overline{x}.$$

Let us again modify the expected transfer to the high type seller so that his utility remains unchanged

$$\Delta \overline{t} = \Delta \overline{x}\, \overline{c},$$

and the expected transfer to the low type seller so that his incentive compatibility constraint is satisfied with equality

$$\Delta \underline{t} - \Delta \underline{x}\, \underline{c} = \Delta \overline{t} - \Delta \overline{x} \underline{c}.$$

Substituting the previous values of $\Delta \underline{x}$ and $\Delta \overline{t}$ into the above equation yields

$$\Delta \underline{t} = \frac{1}{1-\mu} \left((1-\mu)\overline{c} - \underline{c} \right) \Delta \overline{x}.$$

As before these changes make high type sellers indifferent and strictly improve the utility of low type sellers:

$$\Delta \underline{t} - \Delta \underline{x}\, \underline{c} = \left[\frac{1}{1-\mu} \left((1-\mu)\overline{c} - \underline{c} \right) + \frac{\mu}{1-\mu} \underline{c} \right] \Delta \overline{x} = \overline{c} - \underline{c} > 0.$$

They also increase total surplus (while satisfying incentive compatibility and the feasibility constraints imposed by the matching technology) because trades of the good with the lower gains are substituted by trades of the good with the higher gains. Finally, they make buyers weakly better off and therefore constitute a Pareto improvement if

$$\begin{aligned} & \overline{\lambda}^p \left[\Delta \overline{x}\, \overline{v} - \Delta \overline{t} \right] + \underline{\lambda}^p \left[\Delta \underline{x}\, \underline{v} - \Delta \underline{t} \right] \geq 0, \\ \Leftrightarrow \quad & \frac{\underline{\lambda}^p}{1-\mu} \left[\mu(\overline{v} - \underline{v}) - (\overline{c} - \underline{c}) \right] \Delta \overline{x} \geq 0. \end{aligned}$$

The above inequality is satisfied whenever $\mu \geq \frac{\overline{c} - \underline{c}}{\overline{v} - \underline{v}} \in (0, 1)$.

A.8 Proof of Proposition 4.3

Let W^{GM} and W^{BC} denote total surplus in the equilibrium under general mechanisms and the equilibrium under bilateral contracts, respectively. We are interested in the limiting case of $\mu s \to +\infty$, while b and $(1-\mu)s$ are kept finite, implying that $\overline{\lambda}^p$ tends to $+\infty$ and $\underline{\lambda}^p$ is finite.

Consider first the case of general mechanisms. Given $\lim_{\overline{\lambda}^p \to +\infty} \frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) = 0$, the condition $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is always satisfied, meaning that the limiting case falls into the parameter region of Proposition 3.5. The limit of total surplus is thus given by

$$\begin{aligned}\lim_{\overline{\lambda}^p \to +\infty} W^{GM} &= \lim_{\overline{\lambda}^p \to +\infty} b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c})\right], \\ &= b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})\right].\end{aligned}$$

Consider next the case of bilateral contracts. We can first show that as $\overline{\lambda}^p \to +\infty$, the equilibrium fraction of buyers going to the high quality market, γ, tends to one. As shown in the proof of Proposition 4.2 in the Online Appendix, a buyer's profit in the low and high quality market, respectively, is given by

$$\left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c}),$$
$$\left(1-e^{-\frac{\overline{\lambda}^p}{1-\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}).$$

Suppose γ does not tend to one. Then

$$\lim_{\overline{\lambda}^p \to +\infty}\left(\left(1-e^{-\frac{\overline{\lambda}^p}{1-\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c})\right) = -\infty,$$

implying that the indifference condition for buyers cannot be satisfied. Instead we need γ to be a function of $\overline{\lambda}^p$ such that

$$\lim_{\overline{\lambda}^p \to +\infty}\left(\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right) = l \in \mathbb{R},$$

and l such that buyers are indifferent between both markets. Since $\lim_{\overline{\lambda}^p \to +\infty}\gamma(\overline{\lambda}^p) = 1$, the limit of total surplus in the equilibrium with bilateral contracts is then given by

$$\begin{aligned}
\lim_{\overline{\lambda}^p \to +\infty} W^{BC} &= \lim_{\overline{\lambda}^p \to +\infty} b\left[(1-\gamma(\overline{\lambda}^p))\left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right)(\underline{v}-\underline{c}) + \gamma(\overline{\lambda}^p)\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}}\right)(\overline{v}-\overline{c})\right] \\
&= b(\overline{v}-\overline{c}), \\
&> \lim_{\overline{\lambda}^p \to +\infty} W^{GM}.
\end{aligned}$$

□

References

[1] Akerlof, G.A., 1970. The Market for "Lemons": Quality Uncertainty and the Market Mechanism. The Quarterly Journal of Economics, 84(3), 488-500.

[2] Albrecht, J., Gautier, P. and S. Vroman, 2016. Directed Search in the Housing Market. Review of Economic Dynamics, 19, 218-231.

[3] Albrecht, J.W., P.A. Gautier and S.B. Vroman, 2006. Equilibrium Directed Search with Multiple Applications. Review of Economic Studies, 73, 869-891.

[4] Albrecht, J.W., P.A. Gautier and S.B. Vroman, 2014. Efficient Entry in Competing Auctions. American Economic Review, 104(10), 3288-3296.

[5] Bisin, A. and P. Gottardi, 2006. Efficient Competitive Equilibria with Adverse Selection, Journal of Political Economy, 114, 485-516.

[6] Blouin, M.R. and R. Serrano, 2001. A Decentralized Market with Common Values Uncertainty: Non-Steady States. Review of Economic Studie, 68, 323-346.

[7] Burdett, S., S. Shi and R. Wright, 2001. Pricing and Matching with Frictions. Journal of Political Economy, 109, 1060-85.

[8] Cai, X., P. Gautier and R. Wolthoff, 2017a. Search Frictions, Competing Mechanisms and Optimal Market Segmentation. Journal of Economic Theory, 169, 453-473.

[9] Cai, X., P. Gautier and R. Wolthoff, 2017b. Meetings and Mechanisms. Working Paper.

[10] Camargo, B. and B. Lester, 2014. Trading Dynamics in Decentralized Markets with Adverse Selection. Journal of Economic Theory, 153, 534-568.

[11] Chiu, J., and T. V. Koeppl, 2016. Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery. The Review of Economic Studies 83, 969.

[12] Davoodalhosseini, M., 2017. Constrained Efficiency with Adverse Selection and Directed Search. Working Paper.

[13] Delacroix, A. and S. Shi, 2013. Pricing and Signaling with Frictions. Journal of Economic Theory 148, 1301-1332.

[14] De Los Santos, B., Hortacsu, A. and M.R. Wildenbeest, 2012. Testing Models of Consumer Search Using Data on Web Browsing and Purchasing Behavior. American Economic Review, 102(6), 2955-2980.

[15] Dubey, P. and J. Geanakoplos, 2002. Competitive Pooling: Rothschild-Stiglitz reconsidered. Quarterly Journal of Economics, 107(4), 1529-1570.

[16] Eeckhout, J. and P. Kircher, 2010. Sorting vs Screening - Search Frictions and Competing Mechanisms. Journal of Economic Theory, 145, 1354-1385.

[17] Fuchs, W. and A. Skrzypacz, 2013. Costs and Benefits of Dynamic Trading in a Lemons Market. Working Paper.

[18] Gale, D., 1992. A Walrasian Theory of Markets with Adverse Selection. Review of Economic Studies, 59, 229-55.

[19] Galenianos, M. and P. Kircher, 2009, Directed Search with Multiple Job Applications. Journal of Economic Theory, 114(2),445-471.

[20] Guerrieri, V., R. Shimer and R. Wright, 2010. Adverse Selection in Competitive Search Equilibrium. Econometrica, 78(6), 1823-1862.

[21] Inderst, R. and H. Mueller, 1999. Competitive Search Markets with Adverse Selection. Working Paper.

[22] Janssen, M. and R. Roy, 2002. Dynamic trading in a durable good market with asymmetric information. International Economic Review 43(1), 257-282.

[23] Kim, K. and P. Kircher, 2015. Efficient Competition through Cheap Talk: The Case of Competing Auctions. Econometrica 83(5), 1849-1875.

[24] Lester, B., L. Visschers and R. Wolthoff, 2015. Meeting Technologies and Optimal Trading Mechanisms in Competitive Search Markets. Journal of Economic Theory, 155, 1-15.

[25] Lester, B., S. Shourideh, V. Venkateswaran and A. Zetlin-Jones, 2017a. Screening and Adverse Selection in Frictional Markets. Working Paper.

[26] Lester, B., Visschers, L. and R. Wolthoff, 2017b. Competing with Asking Prices. Theoretical Economics, 12, 731-770.

[27] Manelli, A. and D. Vincent, 1995. Optimal Procurement Mechanism. Econometrica, 63, 591-620.

[28] Maurin, V., 2017. Liquidity Fluctuations in Over The Counter Markets. Working Paper.

[29] McAfee, P., 1993. Mechanisms Design by Competing Sellers. Econometrica, 61, 1281-1312.

[30] Moen, E., 1997. Competitive Search Equilibrium. Journal of Political Economy, 105, 385-411.

[31] Moreno, D. and J. Wooders, 2016. Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention. Theoretical Economics, 11, 601-639.

[32] Peters, M., 1997. A Competitive Distribution of Auctions. The Review of Economic Studies, 64(1), 97-123.

[33] Peters, P. and S. Severinov, 1997. Competition Among Sellers Who Offer Auctions Instead of Prices. Journal of Economic Theory, 75(1), 75(1), 141-179.

[34] Rothschild, M. and J. Stiglitz, 1976. Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information. Quarterly Journal of Economics, 90, 629-649.

[35] Samuelson, W., 1984. Bargaining under Asymmetric Information. Econometrica, 52, 995?1005.

[36] Shimer, R., 2005. The Assignment of Workers to Jobs in an Economy with Coordination Frictions. Journal of Political Economy, 2005, 113 (5), 996-1025.

[37] Shi, S., 2001. Frictional Assignment, Part I: Efficiency. Journal of Economic Theory 98, 232-260.

[38] Shi, S., 2002. A Directed Search Model of Inequality with Heterogeneous Skills and Skill-Biased Technology. Review of Economic Studies 69, 467-491.

[39] Wolthoff, R., 2017. Applications and Interviews: Firms' Recruiting Decisions in a Frictional Labor Market. Review of Economic Studies, *forthcoming*.