Competitive Equilibria with Production and Limited Commitment*

Árpád Ábrahám† and Eva Cárceles-Poveda‡

March 9, 2009

Abstract. This paper studies a production economy with aggregate uncertainty where consumers have limited commitment on their financial liabilities. Markets are endogenously incomplete due to the fact that the borrowing constraints are determined endogenously. We first show that, if competitive financial intermediaries are allowed to set the borrowing limits, then the ones that prevent default will be an equilibrium outcome. The equilibrium allocations in this economy are not constrained efficient due to the fact that intermediaries do not internalize the adverse effects of capital on default incentives. We also isolate and quantifiy this new source of inefficiency by comparing the competitive equilibrium allocations to the constrained efficient ones both qualitatively and quantitatively. We tend to observe higher capital accumulation in the competitive equilibrium, implying that agents may enjoy higher (average) welfare in the long run than in the constrained efficient allocation.

Keywords: Enforcement Constraints, Intermediation, Risk Sharing, Capital Accumulation.

JEL Classification: D52, E23

*This paper has been circulating previously under the title of "Risk Sharing Under Limited Commitment". The authors thank the participants of the SED (2007), the AEA (2006) and the Midwest Macroeconomic meetings (2006) and seminar participants at ITAM, EUI, Institute for Advanced Studies and Queen's for their comments.

† *Correspondence:* Department of Economics, University of Rochester, Harkness Hall, P.O.Box 270156, Rochester NY 14627. *E-mail:* aabraha2@mail.rochester.edu.

‡ *Correspondence:* Department of Economics, State University of New York, Stony Brook NY-11794-4384. *Email:* ecarcelespov@notes.cc.sunysb.edu. *Web:* http://ms.cc.sunysb.edu/~ecarcelespov/.

1. Introduction

This paper studies a production economy with aggregate and idiosyncratic uncertainty in which consumers have limited commitment on their financial liabilities. Even though households can trade in a complete set of state contingent claims, markets are endogenously incomplete due to the presence of endogenous borrowing limits, which are determined at the level which makes agents indifferent between repaying their debt or going into financial autarky.

An environment with complete financial markets and endogenous borrowing constraints was first studied by Alvarez and Jermann (2000). Apart from the fact that the previous authors only analyze endowment economies, neither their model nor the subsequent literature has provided micro foundations for how the endogenous limits arise in equilibrium. In the present paper, we do this by introducing a financial intermediation sector with two distinct roles. First, it intermediates between households and the representative firm by collecting funds from the household sector, transforming it into capital and renting it to the production sector. Second, it is allowed to set the borrowing constraints on households, a new feature that has the following important consequence. Regardless of whether the framework is a production or an endowment economy, competition in the intermediation sector implies that the endogenous borrowing constraints which (just) prevent default arise as a (Nash) equilibrium outcome when intermediaries are allowed to set them. Moreover, if the limits are binding in equilibrium, they constitute the unique (Nash) equilibrium outcome. One attractive feature of this result is that these limits do not require any governmental intervention and, in this sense, are self-enforcing. We consider this an important contribution of the present paper.

One of the key questions analyzed in the literature with complete markets and limited commitment is whether a market arrangement with endogenous borrowing constraints that prevent default in equilibrium is constrained efficient. In endowment economies, Alvarez and Jermann (2000) show that this is the case. However, in economies with endogenous production and financial intermediaries, Ábrahám and Cárceles-Poveda (2006) show that the equilibrium with endogenous borrowing constraints is inefficient due to the fact that the value of autarky and thus the incentives to default depend on the aggregate capital stock. In addition, the authors show that a decentralization of the constrained efficient allocations as competitive equilibria with endogenous borrowing limits becomes possible if one also imposes an upper limit on the intermediaries' capital holdings.[1] Since the intermediaries in our economy are not subject to accumulation constraints, it becomes clear that the equilibrium concept studied in the present paper is not constrained efficient. In spite of this, several im-

[1] In a similar economy, a different decentralization with capital income taxes is provided by Chien and Lee (2005). Moreover, Kehoe and Perri (2002b and 2004) provide another decentralization of the constrained efficient economies in a two sector model in which agents are interpreted as countries.

portant reasons provide us with a strong motivation for studying such an equilibrium. First, we are not aware of any empirical evidence on capital accumulation constraints in the data. Second, if intermediaries are subject to accumulation constraints, we show that the endogenous borrowing constraints that just prevent default cannot arise as an equilibrium outcome. The main reason is that intermediaries typically make strictly positive profits in the presence of accumulation constraints, and this is not compatible with perfect competition and free entry. Last, several authors have studied competitive equilibria with production and endogenous borrowing limits that just prevent default for quantitative analysis (see for example Cordoba (2008) and Krueger and Perri (2006)). These papers, however, do not discuss the sustainability of these limits or the relationship between these equilibria and the constrained efficient allocations.

In the present paper, we provide a characterization of the equilibrium in these non-optimal economies that has the following important implications. First, while the computation of competitive equilibria is potentially very demanding, we show that the equilibrium allocations solve almost the same system of equations as the constrained efficient ones. This implies that computing our more empirically plausible competitive equilibrium does not require any extra burden as compared to the relatively easy computation of the constrained efficient solution. Second, our characterization isolates a particular form of inefficiency of limited commitment models that only arises with (endogenous) capital accumulation. As mentioned earlier, this occurs because the value of autarky is positively related to the aggregate capital stock through the dependence of wages (which are the only source of income during autarky) on aggregate capital. This effect cannot be internalized by the endogenous borrowing constraints alone and it is always present whenever the participation (borrowing) constraints are binding for some agent. Third, the proof of the characterization result establishes that the competitive equilibrium allocation we study is equivalent to the equilibrium studied by Kehoe and Levine (1993). This equilibrium concept assumes that agents trade in state-contingent claims at period zero, while their consumption plans have to satisfy participation constraints for every future contingency. Since our equilibrium is inefficient, an important consequence of this is that a Kehoe-Levine equilibrium is inefficient in production economies as well.

After characterizing the competitive equilibrium, the paper compares quantitatively the allocations to the constrained efficient ones in a framework that directly extends the economies studied by Thomas and Worrall (1988), Kocherlakota (1996) and Alvarez and Jermann (2001) to endogenous production. These authors analyze endowment economies with limited commitment and two types of agents that are subject to negatively correlated idiosyncratic income shocks. These assumptions are particularly attractive for illustrating the different effects of limited commitment in the presence of endogenous production, since it is relatively easy to study the effect of aggregate shocks and compute the transition dynamics. Further, this exercise is critical for exploring the qualitative and quantitative effects of the fact that the autarky effects are not internalized.[2]

One of our main findings is that the difference between the equilibrium and constrained efficient allocation is in general relatively small. In particular, we find that the two economies exhibit perfect risk sharing in the long run with the benchmark calibration, assuming standard values for the capital share and impatience level. These results are in line with Cordoba (2008) and Krueger and Perri (2006), who both find extensive risk sharing in models with capital accumulation and endogenous borrowing limits.[3] In addition to this, we find that important differences between the constrained efficient and the equilibrium allocations arise in the short run. First, as expected, the competitive equilibrium accumulates more capital because of the adverse effect of capital on default incentives, which is not internalized by intermediaries. Second, for the same reason, the constrained efficient economy has a bigger range of initial wealth distributions under which full risk sharing is supported. Finally, a more surprising result is that, although agents can enjoy more risk sharing in the constrained efficient allocation, the fact that capital accumulation is lower (either only along the transition or also in the long run with our alternative parametrization) affects their future utility negatively. We find that this last effect dominates for the more wealthy agents, since less risk sharing reduces their utility to a smaller extent. Given this, the allocation of the (inefficient) competitive equilibrium is not Pareto dominated by the constrained efficient allocation.

Next, we study the sensitivity of our results to alternative model formulations. First, we modify the autarky penalties by allowing agents to save in physical capital after default. This modification can potentially have important qualitative implications as, in this case, higher capital does not necessarily increase the value of autarky. Even though it increases wages, it also reduces the interest rate. Nevertheless, none of the qualitative findings described above are altered, although less risk sharing is obviously supported in this case. This also implies that the interest rate effect is quantitatively less important than the wage effect. Second, we choose a different calibration assuming that agents are more impatient and a lower weight of capital income in their total income. In contrast with the benchmark case, this parametrization, which is more similar to the one used by Alvarez and Jermann (2001), implies that the long run equilibrium allocations are not characterized by perfect risk sharing. As in exchange economies, this result shows that the extent of long run risk sharing depends

[2]Similar results can be obtained in empirically more plausible settings with a continuum of types. However, the study of transitional dynamics and/or aggregate shocks in such an environment would require significantly higher computational costs.

[3]On the other hand, our quantitative results are in contrast to the findings of the two country economy with capital accumulation and limited commitment studied by Kehoe and Perri (2002a), where imperfect risk sharing arises in the long run. Ábrahám and Cárceles-Poveda (2006) discuss in detail the key differences between the two setups.

crucially on the calibration. In particular, there always exists a level of patience above which perfect risk sharing is the long run outcome. Finally, we find that the short run differences that we have described above also hold in the long run. In particular, capital accumulation in the stationary distribution tends to be higher in the competitive equilibrium. More surprisingly, we find that the competitive equilibrium actually experiences a higher expected (average) welfare in the stationary distribution due to the higher aggregate capital.

The paper is organized as follows. Section 2 introduces the model economy and describes the constrained efficient allocations. Section 3 discusses and characterizes the competitive equilibrium with endogenous borrowing limits and financial intermediaries. In addition, Section 4 compares the competitive equilibria to the constrained efficient allocations quantitatively and Section 5 summarizes and concludes.

2. The Economy

We consider an infinite horizon economy with production, aggregate uncertainty, idiosyncratic risk and participation constraints. These constraints assume that the continuation utility derived from any allocation has to be at least as high as the continuation utility from the outside option, which is assumed to be financial autarky.[4]

Time is discrete and indexed by $t = 0, 1, 2....$ Further, the resolution of uncertainty is represented by an information structure or event-tree N. Each node $s^t \in N$, summarizing the history until date t, has a finite number of immediate successors, denoted by $s^{t+1}|s^t$. We use the notation $s^r|s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t. Further, with the exception of the unique root node s^0 at $t = 0$, each node has a unique predecessor, denoted by s^{t-1}. The probability of s^t as of period 0 is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$. Moreover, $\pi(s^r|s^t)$ represents the conditional probability of s^r given s^t. For notational convenience, we let $\{x\} = \{x(s^t)\}_{s^t \in N}$ represent the entire state-contingent sequence of any variable x throughout the paper.

The economy is populated by a finite number of agent types that are indexed by $i \in I$, with a continuum of identical consumers within each type. Households have additively separable preferences over sequences of consumption $\{c_i\}$ of the form:

$$U(\{c_i\}) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t u\left(c_i\left(s^t\right)\right) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_i\left(s^t\right)\right), \tag{1}$$

where $\beta \in (0, 1)$ is the subjective discount factor and E_0 denotes the expectation conditional on information at date $t = 0$. The period utility function u is strictly increasing, strictly concave, unbounded below and continuously differentiable, with $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$.

[4]Our model extends the economies in Kocherlakota (1996) and Alvarez and Jermann (2000) to a context with endogenous production.

At each date-state s^t, households are subject to a stochastic labour endowment $\epsilon_i(s^t)$ that follows a stationary Markov chain with N_ϵ possible values. Households supply labor inelastically and the sum of their labour endowments is equal to the aggregate labor supply $L(s^t) = \sum_{i\in I} \epsilon_i(s^t) \in \mathbb{R}_{++}$. Each period, households are also subject to participation constraints of the form:

$$\sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi\left(s^r\right) u\left(c_i\left(s^r\right)\right) \geq V(S_i\left(s^t\right)) \text{ for all } i \in I \text{ and } s^t, \tag{2}$$

where $V(S_i(s^t))$ is the outside option, $S_i(s^t) = (\epsilon_i(s^t); \epsilon_{-i}(s^t), z(s^t), K(s^{t-1}))$, $K(s^t) \in \mathbb{R}_{++}$ is the aggregate capital stock and $\epsilon_{-i} = (\epsilon_i)_{i\in I\setminus i}$.

At each node s^t, a single consumption good $y(s^t) \in \mathbb{R}_+$ is produced with aggregate capital and labor according to the technology:

$$y(s^t) = f(z(s^t), K(s^{t-1}), L(s^t)), \tag{3}$$

where $z(s^t) \in \mathbb{R}_{++}$ is a productivity shock that follows a stationary Markov chain with N_z possible values. Given z, the production function $f(z,\cdot,\cdot) : \mathbb{R}^2_+ \to \mathbb{R}_+$ is assumed to be continuously differentiable on the interior of its domain, strictly increasing, strictly concave in K, and homogeneous of degree one in the two arguments. Moreover, we assume that $f_{LK}(z,K,L) > 0$, $\lim_{K\to 0} f_K(z,K,L) = \infty$ and $\lim_{K\to\infty} f_K(z,K,L) = 0$ for all $K > 0$ and $L > 0$. Capital depreciates at a constant rate δ and we define total output as the sum of output minus the undepreciated capital:

$$F\left(s^t\right) = y\left(s^t\right) + (1-\delta)K(s^{t-1}). \tag{4}$$

The resource constraint of the economy at s^t can then be written as:

$$\sum_{i\in I} c_i(s^t) + K(s^t) = F\left(s^t\right). \tag{5}$$

2.1. Constrained Efficient Allocations. The constrained efficient allocations of the economy described above are characterized in detail by Ábrahám and Cárceles-Poveda (2006). We therefore just provide the first-order conditions below.[5] The first optimality condition is given by:

$$\frac{u'\left(c_i(s^t)\right)}{u'\left(c_j(s^t)\right)} = \frac{\left(1+v_j(s^t)\right)}{\left(1+v_i(s^t)\right)} \frac{u'\left(c_i(s^{t-1})\right)}{u'\left(c_j(s^{t-1})\right)} \text{ for all } s^t \text{ and } i,j \in I. \tag{6}$$

where v_i is a non-negative multiplier that is strictly positive only if the participation constraint of type i is binding. As usual in models with endogenously incomplete

[5] The first-order conditions for this problem are only necessary but not sufficient in general. For a detailed discussion of this issue see Ábrahám and Cárceles-Poveda (2006). A later section that presents the numerical results discusses further how to obtain these conditions.

markets, condition (6) implies that the relative consumption of any two types is determined by the ratio of their time varying Pareto weights, which is represented by the right hand side of the previous equation. The optimality condition that determines the aggregate capital stock is given by:

$$\begin{aligned} 1 &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} (1 + v_i(s^{t+1})) F_K\left(s^{t+1}\right) \right\} \\ &\quad - \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{j \in I} \frac{v_j(s^{t+1})}{u'\left(c_j(s^t)\right)} V_K(S_j\left(s^{t+1}\right)) \right\} \text{ for any } i \in I \text{ and for all } s^t. \end{aligned} \tag{7}$$

where the terms F_K and V_K represent the derivatives of total output F and of the outside option value V with respect to the aggregate capital K. As reflected by condition (7), the presence of binding enforcement constraints introduces two additional effects on the inter-temporal allocation of consumption and capital. First, it increases the planner's marginal rate of substitution between period t and $t+1$ goods, an effect that is reflected by *the presence of* $v_i\left(s^{t+1}\right)$ on the right hand side of the equation. Second, it increases the value of financial autarky, an effect that is reflected by *the autarky effects* V_K on the second part of the right hand side of the equation.

3. Competitive Equilibria

This section extends two different competitive equilibrium concepts to a context with endogenous production. Both types of equilibria have been shown to decentralize the constrained efficient allocations with participation constraints and financial autarky as an outside option in exchange economies. The first one was proposed by Alvarez and Jermann (2000) and the second one by Kehoe and Levine (1993). Whereas the equilibrium proposed by Alvarez and Jermann assumes sequential trade in one period ahead contingent claims subject to endogenous solvency constraints, the equilibrium proposed by Kehoe and Levine imposes the participation constraints as direct restrictions on the consumption possibility sets of consumers. In addition, it assumes an Arrow Debreu market structure with trade at period zero.

In the two settings, we assume that the economy is populated by a representative firm that operates the production technology and by a risk neutral and competitive financial intermediation sector that operates the investment technology. Since we will consider only symmetric equilibria where all intermediaries hold the same portfolio, we focus on the representative intermediary.

3.1. Competitive Equilibrium with Solvency Constraints. This section defines a competitive equilibrium with complete markets and endogenous borrowing limits in the spirit of Alvarez and Jermann (2000).

Each period, the representative firm rents labor from the households and physical capital from the intermediary to maximize period profits:

$$\max_{K(s^{t-1}),L(s^t)} f(z(s^t), K(s^{t-1}), L(s^t)) - w\left(s^t\right) L(s^t) - r(s^t)K(s^{t-1}).$$

Optimality implies that the equilibrium factor prices are given by:

$$w(s^t) = f_L(s^t) \equiv f_L(z(s^t), K(s^{t-1}), L(s^t)) \ \forall s^t \tag{8}$$

$$r(s^t) = f_K(s^t) \equiv f_K(z(s^t), K(s^{t-1}), L(s^t)) \ \forall s^t. \tag{9}$$

The representative intermediary lives for two periods.[6] An intermediary that is born at node s^t first decides how much capital $k(s^t)$ to purchase. The capital is rented to the firm, earning a rental revenue of $r(s^{t+1})k(s^t)$ and a liquidation value of $(1-\delta)k(s^t)$ the following period. To finance the capital purchases, the intermediary sells the future consumption goods in the spot market for one period ahead contingent claims, which are traded at price $q(s^{t+1}|s^t)$. At s^t, the intermediary solves:

$$\max_{k(s^t)} \left\{ -k(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) \left[r(s^{t+1}) + 1 - \delta\right] k(s^t) \right\}.$$

Optimality implies that the intermediary makes zero profits:

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + 1 - \delta] \ \forall s^t. \tag{10}$$

At each s^t, households can trade in a complete set of state contingent claims to one period ahead consumption. They solve the following problem:

$$\max_{\left\{c_i, a_i'\right\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t u\left(c_i\left(s^t\right)\right) \text{ s.t.}$$

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i\left(s^{t+1}\right) \leq a_i\left(s^t\right) + w\left(s^t\right)\epsilon_i\left(s^t\right) \tag{11}$$

$$a_i\left(s^{t+1}\right) \geq A_i\left(s^{t+1}\right). \tag{12}$$

Equation (11) is the budget constraint, where $a_i(s^{t+1})$ is the amount of state contingent claims held by $i \in I$ at the end of period t. Note that market clearing for the state contingent securities requires that the debt issued by the intermediaries matches the demand of the households, that is, $\sum_{i\in I} a_i\left(s^{t+1}\right) = [r(s^{t+1}) + (1-\delta)]K\left(s^t\right)$. Further, equation (12) reflects that the state contingent claims are subject to a borrowing

[6] This assumption implies that intermediaries solve a static problem and consequently helps us to avoid the shareholder disagreement problem that typically arises with incomplete markets. (See Cárceles-Poveda and Coen-Pirani, 2008, for further discussion of this issue.) However, due to competition, intermediaries make zero profits every period, implying that the assumption is without a loss of generality.

constraint of $A_i(s^{t+1})$. The equilibrium determination of these limits will be discussed later on.

If $\zeta_i(s^{t+1}) \geq 0$ is the multiplier on this constraint, the first order conditions with respect to $a_i(s^{t+1})$ imply that:

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t)\left\{\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}\right\} + \frac{\zeta_i(s^{t+1})}{u'\left(c_i(s^t)\right)} \;\forall s^{t+1}|s^t. \tag{13}$$

Finally, the transversality condition in terms of wealth is given by:

$$\lim_{t\to\infty}\sum_{s^t}\beta^t\pi(s^t)u'\left(c_i(s^t)\right)\left[a_i(s^t) - A_i(s^t)\right] \leq 0 \;\forall s^t. \tag{14}$$

Definition 1. *A competitive equilibrium with solvency constraints $\{A_i\}_{i\in I}$, and initial conditions $K(s^{-1})$ and $\{a_i(s^0)\}_{i\in I}$ is a vector of allocations $\left\{(c_i, a_i)_{i\in I}, k, K\right\}$ and prices $\{w, r, q\}$ such that (i) given prices, $\{c_i, a_i\}$ solves the problem for each household $i \in I$; (ii) the factor prices $\{w, r\}$ satisfy the optimality conditions of the firm; (iii) k satisfies the optimality condition of the intermediary; (iv) all markets clear, i.e., for all $s^t \in N$, $k(s^t) = K(s^t)$, $\sum_{i\in I} a_i(s^{t+1}) = [r(s^{t+1}) + 1 - \delta]K(s^t)$, $\sum_{i\in I}\epsilon_i(s^t) = L(s^t)$ and $\sum_{i\in I} c_i(s^t) + K(s^t) = F(s^t)$.*

As stated in the previous section, households have an outside option of $V(S_i(s^t))$. Following the existing literature, we assume that households can leave the risk sharing arrangement at any date-state to go to financial autarky. In this case, they will only be able to consume their labour income, while they are excluded from financial markets forever.[7] To take this into account, we impose endogenous borrowing limits, in the sense that a looser limit would imply that an agent with that level of debt prefers to leave the trading arrangement. To define these borrowing constraints, the value of the trading arrangement can be written recursively as follows:

$$W(a_i(s^t), S_i(s^t)) = u\left(c_i\left(s^t\right)\right) + \beta\sum_{s^{t+1}|s^t}\pi(s^{t+1}|s^t)W(a_i(s^{t+1}), S_i(s^{t+1})). \tag{15}$$

Definition 2. *The endogenous borrowing constraints $\{A_i\}_{i\in I}$ satisfy the following condition for all $i \in I$ and all nodes $s^t \in N$:*

$$W(A_i(s^t), S_i(s^t)) = V(S_i(s^t)), \tag{16}$$

where the value of the outside option at s^t is given by:

$$V(S_i(s^t)) = \sum_{r=t}^{\infty}\sum_{s^r|s^t}\beta^{r-t}\pi\left(s^r|s^t\right)u\left(w\left(s^r\right)\epsilon_i(s^r)\right). \tag{17}$$

It is important to note that the value of staying in the trading arrangement W is strictly increasing in asset wealth, whereas the autarky value V is not a function

[7] A different outside option under which households are excluded from trade in Arrow securities but can still save by accumulating physical capital is considered later on.

of $a_i(s^t)$. This implies that the limits defined by (16) exist and are unique under our assumptions on the utility function. Moreover, since $W(0, S_i(s^t)) \geq V(S_i(s^t))$ and W is increasing in a_i, equation (16) implies that $A_i(s^t) \leq 0$. Intuitively, no agent would default with a positive level of wealth, since he could then afford a higher current consumption than in autarky and at least as high of a life-time utility as in autarky from next period on.

Micro foundations for the endogenous limits. So far, we have assumed that the limits defined by (16) are taken as given by the intermediaries and the agents. In what follows, we provide further micro foundations for these borrowing constraints by allowing the intermediaries to set both the limits and the Arrow security prices. We will consider symmetric Nash equilibria in this setting. In particular, we check whether an intermediary has gains from deviating from a particular strategy (limits and prices) while all the other ones stick to the same (equilibrium) strategies. The deviations we will consider typically involve more lending than the one allowed by the postulated equilibrium limits, potentially combined with a different price. Proposition 1 shows that the above borrowing limits will arise as an equilibrium outcome if the intermediaries are allowed to set them. The proof of this and of all remaining propositions are relegated to Appendix 1.

Proposition 1. *(i) The competitive equilibrium with endogenous borrowing constraints remains a competitive (Nash) equilibrium if the intermediaries are allowed to set the borrowing limits. (ii) No symmetric competitive (Nash) equilibrium exists with equilibrium default. (iii) No competitive (Nash) equilibrium with binding borrowing constraint that are tighter than the endogenous limits defined by (16).*

The previous proposition shows first that no intermediary has incentives to loosen or tighten the endogenous limits individually, since these deviations are not profitable. The proof of the first part is intuitive. On the one hand, intermediaries cannot break even with looser limits regardless of the price they charge, since agents will default for sure with higher debt levels. On the other hand, since the intermediaries will make zero profits with any limits that do not allow for default, the intermediaries have no incentives to tighten them either.

The second part of the proposition implies that no symmetric equilibrium exists where some or all the relevant limits (the ones that bind in equilibrium) are looser than the ones dictated by (16). This result is due to the fact that, if there was default, the intermediaries would be able to increase their profits by not buying Arrow securities from households with a positive probability of default next period.

The third part states that no equilibrium exists with tighter limits either. Intuitively, if some binding limits were any tighter, then any intermediary would be able to make some positive profits by offering slightly looser limits (so that they are still tighter than the limits defined by (16)). Whereas this would keep lending still risk-free, this intermediary could charge a slightly higher interest rate as agents would like to borrow more. Note that this is guaranteed by the fact that agents are actually borrowing constrained under the original asset prices q. Hence, there would be a higher interest rate (lower q) such that they are still willing to borrow more under the new prices.[8]

Finally, note that Proposition 1 implies that (at least among the symmetric equilibria) the equilibrium with endogenous borrowing constraints is unique in the following sense: these are the only possible limits for those states/agents in which a particular agent's borrowing constraint is binding.

Note that this proposition does not rely on the fact that we have a production economy. In fact, the proof would go through in an almost identical way if we consider an exchange economy. Except for the fact that we would have $r(s^t) = \delta = 0$ and $k(s^t) = 0$ for all s^t in that case, all the steps of the proof would go through.

Second, notice also that the proof critically relies on the fact that intermediaries make zero profits in equilibrium. In Ábrahám and Cárceles-Poveda (2006), we impose capital accumulation constraints on intermediaries and show that competitive equilibria become efficient with some carefully chosen accumulation constraints. However, that also implies that intermediaries will make positive profits in equilibrium and the above argument would not go through. In this sense, the competitive equilibrium which is constrained efficient is not self-enforcing, since the endogenous limits satisfying (16) would not arise as an equilibrium outcome if the intermediation sector has the ability to set the limits. The fact that this competitive equilibrium has stronger micro foundations provides a further motivation for studying it despite the fact that it is inefficient.

3.2. Competitive Equilibrium with Participation Constraints. This section defines a competitive equilibrium with complete markets and participation constraints following Kehoe and Levine (1993). Given that securities are only traded at period zero, we assume that the representative firm and the representative financial intermediary are infinitely lived.[9]

The representative firm rents labor from the households and physical capital from the intermediary to maximize profits, which are sold forward in the state contingent markets. The firm maximizes:

$$\max_{\{K(s^{t-1}),L(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \left[f(z(s^t), K(s^{t-1}), L(s^t)) - w\left(s^t\right) L(s^t) - r(s^t)K(s^{t-1}) \right].$$

[8] It is important to note that Proposition 1 implies the existence of symmetric equilibria with no default borrowing constraints as well as the non-existence of symmetric equilibria with equilibrium default. However, we cannot rule out non-symmetric equilibria where default can arise in equilibrium.

[9] Note that although both the firm and the intermediary live for ever, they practically solve a static problem at time zero taking the prices for state and time contingent goods as given.

where $Q(s^t|s^0)$ is the price at time zero of consumption at period t, contingent on history s^t. Profit maximization implies that the equilibrium factor prices are given by:

$$w(s^t) = f_L(s^t) \equiv f_L(z(s^t), K(s^{t-1}), L(s^t)) \ \forall s^t \tag{18}$$

$$r(s^t) = f_K(s^t) \equiv f_K(z(s^t), K(s^{t-1}), L(s^t)) \ \forall s^t. \tag{19}$$

The representative intermediary decides how much capital $k(s^t)$ to purchase from households. The capital is rented to the firm, earning a rental revenue of $r(s^{t+1})k(s^t)$ and a liquidation value of $(1-\delta)k(s^t)$ the following period. The revenues net of capital purchases are sold in the state contingent markets for next period goods. The intermediary maximizes:

$$\max_{\{k(s^t)\}} \left\{ \sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \left[\left(r(s^t) + 1 - \delta \right) k(s^{t-1}) - k\left(s^t\right) \right] \right\}.$$

Optimality implies that intermediaries make zero profits:

$$1 = \sum_{s^{t+1}|s^t} \frac{Q(s^{t+1}|s^0)}{Q(s^t|s^0)} [r(s^{t+1}) + 1 - \delta] \ \forall s^t, t. \tag{20}$$

Finally, households solve the following problem:

$$\max_{\{c_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u\left(c_i\left(s^t\right)\right) \text{ s.t.}$$

$$\sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \left[c_i(s^t) - w\left(s^t\right) \epsilon_i\left(s^t\right) \right] \ \leq \ a_i(s^0) \tag{21}$$

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \ \geq \ V(S_i\left(s^t\right)) \ \forall s^t, t. \tag{22}$$

Equation (21) is the consolidated budget constraint in an Arrow Debreu market structure with trade in period zero, whereas equation (22) illustrates that participation constraints are imposed as direct restrictions on the consumption sets for every contingency s^t and time period t. Given the presence of the latter constraints, standard dynamic programming is not applicable to the previous problem. However, we can use the recursive contracts approach of Marcet and Marimon (1999) to rewrite the Lagrangian recursively as follows:

$$\inf_{\{\gamma_i\}} \sup_{\{c_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t \left\{ u(c_i(s^t))(1 + \mu_i(s^t)) - \gamma_i(s^t) V(S_i\left(s^t\right)) \right\}$$
$$+ \eta_i \left[a_i\left(s^0\right) - \sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \left[c_i(s^t) - w\left(s^t\right) \epsilon_i\left(s^t\right) \right] \right].$$

where $\beta^t \gamma_i(s^t)$ is the Lagrange multiplier of the time t participation constraint for household $i \in I$ and $\mu_i(s^t)$ is a co-state variable that is equal to:

$$\mu_i(s^t) = \mu_i(s^{t-1}) + \gamma_i(s^t),\ \mu_i(s^{-1}) = 0 \text{ for } i \in I. \tag{23}$$

The first order conditions imply that the Arrow prices are given by:

$$Q(s^t|s^0) = \frac{\pi(s^t)\beta^t u'(c_i(s^t))(1 + \mu_i(s^t))}{\eta_i}. \tag{24}$$

Definition 3. *A competitive equilibrium with participation constraints and initial conditions $K(s^{-1})$ and $\{a_i(s^0)\}_{i \in I}$ is a vector of allocations $\left\{(c_i)_{i \in I}, k, K\right\}$ and prices $\{w, r, Q\}$ such that (i) given prices, $\{c_i\}$ solves the problem for each household $i \in I$; (ii) the factor prices $\{w, r\}$ satisfy the optimality conditions of the firm; (iii) k satisfies the optimality condition of the intermediary; (iv) all markets clear, i.e., for all $s^t \in N$, $k(s^t) = K(s^t)$, $\sum_{i \in I} \epsilon_i(s^t) = L(s^t)$ and $\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t)$.*

4. Characterization of the Competitive Equilibria

This section characterizes the competitive equilibria described in section 3. We first show that a competitive equilibrium with solvency constraints is also a competitive equilibrium with participation constraints and vice versa. Ábrahám and Cárceles-Poveda (2006) show that the competitive equilibrium with solvency constraints is suboptimal. Given this, our equivalence results imply that the equilibrium concept proposed by Kehoe and Levine (1993) is also suboptimal in the presence of endogenous production. We then identify the key source of the inefficiency and we provide an additional characterization of the two competitive equilibrium that makes their computation easy.

As in the literature, we focus on allocations that have high implied interest rates, in the sense that their present value is finite.[10] More precisely, we say that an allocation $\{c\} \equiv \left\{\sum_{i \in I} c_i\right\}$ has high implied interest rates if:

$$\sum_{t=0}^{\infty} \sum_{s^t} Q^p(s^t|s^0) c(s^t) < \infty \tag{25}$$

where the pricing function $\{Q^p\}$ is defined as:

$$Q^p(s^t|s^0) = q^p(s^t|s^{t-1}) q^p(s^{t-1}|s^{t-2}) ... q^p(s^1|s^0). \tag{26}$$

$$q^p(s^{t+1}|s^t) = \max_{i \in I} \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} \tag{27}$$

Propositions 2 and 3 state our main equivalence results.

[10] This assumption is not very restrictive in the present setting, since it will be satisfied whenever consumption is bounded away from zero.

Proposition 2. *Let* $\left\{(c_i)_{i\in I}, K, k, Q, w, r\right\}$ *be a competitive equilibrium with participation constraints where* $\{c\} = \sum_i \{c_i\}$ *has high implied interest rates. Then, it is possible to find prices* $\{q\}$ *and asset holdings* $\left\{(a_i)_{i\in I}\right\}$ *such that* $\left\{(c_i, a_i)_{i\in I}, K, k, q, w, r\right\}$ *is a competitive equilibrium with endogenous borrowing constraints.*

Proposition 3. *Let* $\left\{(c_i, a_i)_{i\in I}, K, k, q, r, w\right\}$ *be a competitive equilibrium with endogenous borrowing constraints*$\{A_i\}_{i\in I}$*. Then, there exists prices* $\{Q\}$ *so that* $\left\{(c_i)_{i\in I}, K, k, Q, w, r\right\}$ *is a competitive equilibrium equilibrium with participation constraints where* $c = \sum_i c_i$ *has high implied interest rates.*

Propositions 2 and 3 show that a competitive equilibrium with endogenous borrowing constraints is equivalent to a competitive equilibrium with participation constraints.[11] To show the equivalence of the equilibria, we have used the fact that the optimality condition for the intermediaries in the two settings can be written as follows:

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\beta \max_{i\in I} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} F_K\left(s^{t+1}\right). \tag{28}$$

This can be easily seen by noting that

$$q(s^{t+1}|s^t) = \frac{Q\left(s^{t+1}|s^0\right)}{Q\left(s^t|s^0\right)} = \beta\pi(s^{t+1}|s^t) \max_{i\in I} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\}.$$

An important implication of equation (28) is that the two competitive equilibria are suboptimal whenever any of the solvency or the participation constraints are binding, since (7) and (28) cannot be satisfied by the same allocations. The inefficiency of both competitive equilibria is due to the (externality) effect of capital on the value of autarky, which the intermediaries do not internalize. Propositions 2 and 3 implies that the same inefficiency arises in both the competitive equilibrium with participation constraints and with endogenous borrowing constraints in the presence of endogenous production.

In the next section, we analyze the consequences of this inefficiency quantitatively. Before doing this, we provide an additional characterization of the two competitive equilibria that will make their computation easy. In particular, we show that the allocations from the competitive equilibrium with solvency constraints (and hence the allocations of the competitive equilibrium with participation constraints) satisfy the same system of equations as the constrained efficient problem except the Euler condition in (7), which is replaced by:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \left(1 + v_i\left(s^{t+1}\right)\right) F_K(s^{t+1}) \right\}. \tag{29}$$

[11] Some parts of the proof of Propositions 2 and 3 follow similar arguments to the ones in **the equivalnce proofs of** Alvarez and Jermann (2000). However, an important difference between our results and theirs is that the previous authors do not have endogenous production sector or a financial intermediation sector. In other words, to prove the equivalence of different allocations, we also need to make sure that the optimality conditions of the firms and intermediaries are satisfied.

This characterization is provided by the following two propositions.

Proposition 4. *Let* $\left\{(c_i)_{i\in I}, K\right\}$ *be a solution to equations (2)-(6) and (29) where* $\{c\} = \sum_{i\in I}\{c_i\}$ *has high implied interest rates. Then, this allocation can be decentralized as a competitive equilibrium with endogenous borrowing constraints.*

Proposition 5. *Let* $\left\{(c_i, a_i)_{i\in I}, K, q, r, w\right\}$ *be a competitive equilibrium with endogenous borrowing constraints* $\{A_i\}_{i\in I}$*. Then* $\left\{(c_i)_{i\in I}, K\right\}$ *is a solution to equations (2)-(6) and (29). Further,* $c = \sum_{i\in I} c_i$ *satisfies the high implied interest rates condition with respect to the price* $Q(s^t|s^0)$ *defined by:*

$$Q(s^t|s^0) = q(s^t|s^{t-1})q(s^{t-1}|s^{t-2})...q(s^1|s^0).$$

Propositions 4 and 5 provide a useful characterization of the competitive equilibria defined earlier, since they show that the equilibrium allocations solve a system of equations that is very similar to the one of the constrained efficient allocation. As noted earlier, the equilibrium allocations are different from the optimal allocation only due to the fact that they ignore the autarky effects. In other words, as opposed to the social planner, the financial intermediaries (or the households) do not internalize the effect of capital accumulation on the agents' autarky valuations.

We believe, that these propositions are particularly important, since they characterize an empirically more plausible competitive equilibrium which can be used to analyze several applied questions where capital accumulation and limited commitment are both important. As an example, one could study consumption and wealth inequality along the growth path, where capital accumulation can play an important role in determining the incentives to default. The computation of competitive equilibrium for this type of non-optimal economies is potentially very demanding. In these cases, one important implication of the above propositions is that computing the equilibrium would not require any extra burden as compared to the relatively easy computation of the optimal solution. This is briefly illustrated in the next section.

5. Quantitative Comparison of the Allocations: An Illustrative Example

This section compares the competitive equilibrium allocations to the constrained efficient allocation numerically. To do this, we focus on the competitive equilibrium with solvency constraints. We first describe the benchmark calibration and the solution method. Next, we discuss the quantitative findings.

5.1. Calibration and Solution Method.

The benchmark parameters are calibrated following the asset pricing and real business cycle literature. The time period is assumed to be one quarter, and the discount factor and depreciation rate are therefore set to $\beta = 0.99$ and $\delta = 0.025$. The first parameter is chosen to generate an annual average interest rate of approximately 4% in the stationary distribution, whereas the second replicates the US average investment to capital ratio during the postwar period.

Concerning the functional forms, we assume that the production function is Cobb-Douglas, with a constant capital share of $\alpha = 0.36$. Further, the utility function of the households is assumed to be $u(c) = \log(c)$. Finally, the exogenous shock processes are assumed to be independent with each other. In particular, the aggregate technology shock follows a two state Markov chain with $z \in \{z_l, z_h\} = \{0.99, 1.01\}$, and its transition matrix is given by:

$$\Pi_z = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}.$$

The aggregate labor supply is constant and we normalize it to 1. As to the idiosyncratic income process, it is assumed to follow a seven state Markov chain. The values and transition matrix of the Markov chain are obtained by using the Tauchen and Hussey (1991) procedure to discretize the following process:

$$\epsilon^{i\prime} = (1 - \psi_\epsilon)\mu_\epsilon + \psi_\epsilon \epsilon^i + u,\ u \sim N(0, \sigma_u^2).$$

The autoregressive and variance parameters are set to $\psi_\epsilon = 0.956$ and $\sigma_u^2 = 0.082$, corresponding to quarterly adjusted estimates from annual idiosyncratic earnings data.[12]

To simplify our computations and to relate to the existing literature, we assume that economy is populated by two agent types. The values for ϵ^1 are then chosen to be symmetric around 0.5 and we assume that $\epsilon^2 = 1 - \epsilon^1$ so that the labor supply is constant. This implies that the idiosyncratic productivity of the two types follows the same process and the shocks are perfectly negatively correlated across the two types.

As to the solution method, Propositions 4 and 5 imply that we can use the same algorithm for both the constrained efficient and the competitive equilibrium allocations. Looking at the system of equations that each allocation solves, and using the fact that shocks are Markovian, it is easy to see that the allocations are recursive in $S_i = (\epsilon_i, \epsilon_{-i}, z, K, \lambda)$, where the variable λ is defined as:

$$\lambda(s^t) = \frac{u'(c_1(s^t))}{u'(c_2(s^t))}$$

Note that λ has a different interpretation depending on the allocation. In the constrained efficient allocation, λ can be interpreted as the time varying relative Pareto weight of type 2 households relative to type 1. In the competitive equilibrium, λ does not have the interpretation of a "temporary" relative Pareto weight in the competitive equilibrium but rather of a measure of relative wealth. To see this, consider the competitive equilibrium with solvency constraints. If we define the Lagrange multipliers

[12] The discretization of this process gives positive values for all the states.

of the budget constraint in (11) by $\pi(s^t)\beta^t\xi_i(s^t)$, we have that:

$$\frac{\xi_1(s^t)}{\xi_2(s^t)} = \frac{u'(c_1(s^t))}{u'(c_2(s^t))} = \lambda(s^t),$$

where the second inequality is a consequence of Proposition 5. The above identity implies that $\lambda(s^t)$ measures the relative wealth of type 2 versus type 1, since the bigger is the asset wealth $a_i(s^t)$, the smaller is $\xi_i(s^t)$, which measures the marginal utility of wealth. Therefore a higher $\lambda(s^t)$ implies that agent 1 has a smaller initial wealth compared to type 2 households.

Apart from the fact that the allocations are recursive in $(\epsilon_i, \epsilon_{-i}, z, K, \lambda)$, the symmetry of the households implies that we only need to include their own idiosyncratic shock in the individual state vector. Further, it is easy to see that $1/\lambda$ measures the relative Pareto weight of a given household type if λ is the relative Pareto weight of the other type. Therefore, $s_1 = [\epsilon, \lambda; z, K]$ implies $s_2 = [1-\epsilon, 1/\lambda; z, K]$. Since the shocks are assumed to be Markovian, the previous set of equations imply that we can describe the optimal allocations in both models by the consumption functions $\{c_i(s_i)\}_{i\in I}$, the multipliers $\{\nu_i(s_i)\}_{i\in I}$ and the laws of motion for the relative wealth $\lambda' = \lambda(s_1)$ and the aggregate capital $K' = K(s_1)$. To solve for both the constrained efficient and the competitive equilibrium allocations, we use a policy iteration algorithm that is described in more detail in the Appendix.[13]

5.2. Quantitative Findings. Our numerical results for this benchmark parametrization are presented in Figures 1 to 6 of Appendix 3. All the optimal policies are conditioned on the low aggregate technology shock $z = 0.99$ and on $K = 38.6$, which is the mean of the stationary distribution of capital, but similar pictures can be obtained for the high technology shock. For expositional convenience, we have plotted the results for only three levels of the labour endowment, where ϵ_1 is the lowest and ϵ_7 is the highest labor endowment. Recall that type 2 households have the highest labor endowment when type 1 households have the lowest. Note also that both types have equal endowments when type 1 households have $\epsilon_4 = 1 - \epsilon_4 = 0.5$. Finally, in all the figures, the competitive equilibrium with solvency constraints is labelled as '*Competitive Equilibrium*', while the constrained efficient allocation is labelled as '*Constrained Efficiency'*.

Figure 1 displays λ' as a function of λ for the three different levels of the idiosyncratic income shocks. The first important observation, based on this figure, is that agents enjoy permanent perfect risk sharing in the long run in both models. To see this, assume first that our initial λ is inside its ergodic set, which is equal to $\lambda \in [0.8368, 1.195]$ and $\lambda \in [0.8366, 1.1953]$ for the models without and with the

[13]This algorithm can be easily extended to a context with a continuum of agents. More details of this extension can be provided by the authors upon request.

savings constraint respectively. As we see on the graph, $\lambda' = \lambda$ inside this region, independently of the labor income shocks. However, this can only happen if neither agent's participation constraint is binding. In addition, the ratio of marginal utilities remains constant over time. The last result, however, is the defining feature of a perfect risk sharing allocation.

Assume now that we start with $\lambda > 2.5$, implying that type 1 households hold significantly lower initial assets, and they are therefore entitled to less consumption than type 2 households. In this case, Figure 1 implies that λ' depends on the idiosyncratic income of the agent, and that it will drop to a new level depending on the shock realization. In particular, the higher the idiosyncratic income is, the lower will be the new level of the relative wealth λ'. This is due to the fact that type 1 agents will then enjoy a higher autarky value and require therefore a higher compensation for staying in the risk sharing arrangement.

Here, it is important to note that, whenever λ jumps, type 1 agents' participation constraint is binding, and this new level of λ' pins down the borrowing constraint of the competitive equilibrium faced by type 1 households in the previous period. This process will go on until the highest income (ϵ^7) is experienced by the type 1 agents. In this case, λ will enter the stationary distribution[14] ($\lambda = 1.195$) and remain constant forever. Thus, agents will enjoy permanent perfect risk sharing from that period on. In addition, a symmetric argument implies that whenever $\lambda < 0.83$, λ will become 0.83 and remain constant forever after finite number of periods. Finally, whereas agents will obtain full insurance in the long-run for any initial wealth distribution, note that the economy may experience movements in consumption and in λ in the short run.

The second important observation is that two economies are qualitatively very similar. As stated above, the long-run behavior is practically identical, in the sense that there is perfect risk sharing in the long run. In addition, if $\lambda(s^0) \in [0.8368, 1.195]$, the long-run allocations will be identical. This is due to the fact that the borrowing constraints will never bind in this case in either of the two economies. Thus, the individual consumptions will be determined by $\lambda(s^0)$ and the capital accumulation will be (unconstrained) efficient. However, if $\lambda(s^0)$ is outside the above interval, the long-run allocations will be somewhat different due to the fact that the bounds of the stationary distribution are slightly different in the two models. As we see, the constrained efficient allocation allows for a slightly wider range of λ (the wealth distribution) where the participation constraints are not binding. As we will see below, this is the consequence of the different capital accumulation pattern in the two economies.

Figure 2 shows the optimal consumption of type 1 households in the two economies

[14] We use the terms ergodic set and the stationary distribution loosely in this paper. Notice, however that we defined these sets as the possible values of λ in the long run. In fact, the initial condition λ_0 will pin down a unique long-run value for the relative wealth, that is, for any given initial value, the long run distribution is degenerate.

as a function of λ for different levels of the labor endowment. Obviously, as the relative wealth of type 1 households decreases (λ increases) their consumption decreases. Also, since we have perfect risk sharing in the stationary distribution, consumption does not depend on the idiosyncratic labour endowment there. For the same reason, the optimal consumption allocations are identical across the two models in this range. Outside the stationary distribution, as expected, consumption is increasing in the labour endowment. We also observe that in the constrained efficient allocation consumption is higher for every λ and ϵ outside the stationary distribution. As explained below, this is the consequence of higher capital accumulation in the competitive equilibrium.

Figure 3 displays the next period's aggregate capital K' as a function of λ and ϵ. Again, aggregate capital is independent of both the wealth distribution and the labour endowments in the stationary distribution, where it is at its efficient level. In contrast, markets are effectively incomplete outside the stationary distribution, where we see a higher capital accumulation. This result is well-documented in models with exogenously incomplete markets (see e.g. Aiyagari (1994) for a model without aggregate uncertainty and Ábrahám and Cárceles-Poveda (2007) for a model with a similar set-up but trade in physical capital only). As reflected by the figure, a similar behavior arises in the present setting. In particular, capital accumulation is higher when the low idiosyncratic labour endowment coincides with low wealth (high λ). This is the case for type 1 households on the upper right corner of the figure and for type 2 households in the upper left corner.

To see why this happens, we can look at Figure 1 and at the Euler equation of the constrained efficient problem. It is clear from Figure 1 that, when type 1 households have a labour endowment of ϵ_7 and low λ (high wealth), the participation constraint of type 2 households is going to be binding in many continuation states ($v_i(s^{t+1}) > 0$). In turn, this implies that the return of investment is higher, and more capital will be accumulated.

In the competitive equilibrium, this is equivalent to an increase of most of the Arrow security prices $q(s^{t+1}|s^t)$, implying that intermediaries have to pay a lower return to the agents and can therefore invest more. This is the only effect in the competitive equilibrium. In contrast, this over accumulation is mitigated by the autarky effects in the constrained efficient allocation. In this case, the planner internalizes that a higher capital will increase the autarky values, leading to a lower capital accumulation than in the competitive equilibrium. In this case, households will also have less incentives to default, since the value of their outside option is lower due to a lower capital accumulation. As a consequence, we obtain perfect risk sharing for a higher range of the wealth distribution (a higher range of λ) in the constrained efficient allocation.

Using the results stated in Proposition 2 in Ábrahám and Cárceles-Poveda (2006), we have also depicted the individual consumptions c_i and the next period capital stock K' as a function of the initial Arrow security holdings a_1 and the same levels of idiosyncratic shocks in Figures 4 and 5.[15] As already documented above, Figure 5 illustrates that capital accumulation is always higher in the economy with no capital accumulation constraints. In particular, capital accumulation is the highest when the low idiosyncratic shock for the type 1 households ϵ_1 is combined with a low level of initial asset holdings a_1, or when the high idiosyncratic shock for the type 1 households ϵ_7 is combined with a high level of initial asset holdings a_1. We also note that the difference between the two economies is significant. In the competitive equilibrium, average investment is 15% more than in the constrained efficient allocation when the lowest wealth coincides with the lowest income. Consequently, consumption will be higher in the constrained efficient allocation, especially with these combinations of idiosyncratic income and initial asset holdings. This is reflected in figure 4. Finally, note that the supported asset distribution is wider for the constrained efficient allocation. This implies that agents are facing tighter borrowing constraints in the competitive equilibria, a fact that is not surprising given that higher capital accumulation increases the incentives to default.

Finally, Figure 6 shows the welfare loss in the competitive equilibrium relative to the constraint efficient allocation in consumption equivalent percentage terms for different initial wealth levels and income shocks. Obviously, welfare is identical across the two economies in the stationary distribution, since the allocations are identical. Outside the stationary distribution, however, agents gain some utility in the competitive equilibrium compared to the constrained efficient allocation if they are relatively wealthy ($a_1 > 30$) and they loose some utility when they are less wealthy ($a_1 < 10$). This can be explained by the following two effects. First, both the equilibrium and the constrained efficient allocations exhibit full risk sharing in the long run, implying that they sustain the same long run level of capital. However, outside the ergodic set (or during the transition towards perfect risk sharing), capital is higher and consumption is lower in the competitive equilibrium. Note that a higher aggregate capital leads to higher wages, an effect that benefits all agents. Second, the competitive equilibrium exhibits less risk sharing in the short run due to the fact that borrowing constraints are tighter (capital accumulation is higher). This hurts everybody but particularly the poor agents, who are more likely to be borrowing constrained. Overall, the fact that capital and wages are higher in the competitive equilibrium dominates the consumption and risk sharing loses for rich agents, whereas the opposite happens with poor agents.

Overall, we conclude that both economies have very similar allocations in the long run (stationary distribution), and they exhibit some important differences in the short run. As we have seen, the model without capital accumulation constraints leads to

[15] In the constrained efficient allocations, asset wealth also includes the ownership of shares in the financial intermediary, since the intermediaries make positive profits. For these calculations, we have assumed that both types hold initially the same amount of shares.

higher short run capital accumulation and consequently to a lower current consumption. A key question is how robust these properties are to some key features of our model and calibration. In order to check this, we have also investigated several variations of the above model and calibration in what follows.

Relaxing the Autarky Punishment. In the first experiment, we allow agents to accumulate physical capital in autarky, increasing the value of the outside option and limiting the scope of risk sharing in both economies. Formally, the autarky value at state-date s^t solves the following problem:

$$V(s^t) \equiv \max_{\{c_i(s^{t+\tau}),\kappa_i(s^{t+\tau})\}_{\tau=0}^{\infty}} \sum_{\tau=0}^{\infty} \sum_{s^{t+\tau}} \pi(s^t)\beta^t u\left(c_i\left(s^{t+\tau}\right)\right) \text{ s.t.}$$

$$c_i(s^{t+\tau}) + \kappa_i\left(s^{t+\tau}\right) \leq w(s^{t+\tau})\epsilon_i(s^{t+\tau}) + r\left(s^{t+\tau}\right)\kappa_i\left(s^{t+\tau-1}\right) \text{ for} \forall \tau \geq 0 \tag{30}$$

$$\kappa_i\left(s^{t+\tau}\right) \geq 0 \text{ for } \forall \tau \geq 0 \text{ and } \kappa_i\left(s^{t-1}\right) \equiv 0. \tag{31}$$

where $\kappa_i\left(s^{s^{t+\tau}}\right)$ represents the individual capital holdings of type $i \in I$ households. Note that the budget constraint in (30) implies that households face (exogenously) incomplete asset markets after default. Further, the first constraint in (31) reflects that households can only save but not borrow (short-sell) physical capital after default. Finally, we assume that they take the aggregate capital accumulation and therefore the current and future prices ($w(s^{t+\tau})$ and $r\left(s^{t+\tau}\right)$) as given. Since we only consider individual (Nash) deviations and there is no default in equilibrium, these expectations are indeed rational. Finally, in this case, it is not obvious ex ante whether higher aggregate capital leads to higher or lower autarky values, because although it increases wages it also reduces the interest rate.

Whereas we obtain a narrower range of λ in the stationary distribution, all the key qualitative findings of our original model are robust to this extension. In particular, we still find a perfect risk sharing in the long-run in both economies, while there is higher capital accumulation and a lower consumption in the short run in the constrained efficient allocation.[16] We can therefore conclude that neither the qualitative differences between the two equilibria nor the long-run perfect risk sharing property is a consequence of the tight autarky penalty that we have assumed in the benchmark model. This also implies that the interest rate effect is quantitatively less important than the wage effect, in determining the overall impact of capital on the autarky value in our setup.

Using Different Parameterizations. To see if our results are robust to different parameter values, we have also studied a significantly different parametrization of the benchmark model. First, it is clear that a lower individual discount factor will make default more attractive in this environment. For this reason, we have set β to

[16] More detailed results are available from the authors upon request.

0.65. This relatively low value of the discount factor was used by Alvarez and Jermann (2001), who study asset pricing implications of limited commitment in an endowment economy. Since this parametrization is more consistent with an annual model, we have also increased δ to 0.1. Second, it is clear that our economy is approaching a pure exchange economy as the one studied by Alvarez and Jermann (2000) as α goes to 0. In addition, the higher α is, the more important capital income becomes for the determination of the agents' consumption. In other words, a lower capital share will make default ceteris paribus more attractive. Given this, we have reduced α to 0.20.[17]

Some of the key results resulting from this parametrization are shown on Figures 7 to 9. As shown by Figure 7, the long-run stationary distribution of λ is not degenerate with the new parameterization, implying that the individual shares of aggregate consumption are fluctuating in the long run. First, this shows that the full risk sharing result obtained with the benchmark parametrization is due to the specific parameter values we have chosen before. Second, our results illustrate that the qualitative differences between the two allocations (competitive equilibrium and constrained efficient) remain the same with the new parameterization. In particular, the competitive equilibrium is accumulating more capital, whereas the constrained efficient economy does not Pareto dominate the competitive equilibrium. Since these economies do not exhibit full risk sharing in the long run, we can also study the differences between the two equilibria in the stationary distribution.

These observations are also related to the findings in Thomas and Worrall (1988) and Kocherlakota (1996) (for a textbook treatment of these papers, see Ljungqvist and Sargent, 2004, Chapter 20). These authors study endowment economies with two agents under limited commitment. Thus, the two agent economy we simulate is an extension of their framework to production and capital accumulation.

First, one of their main findings is that constrained efficient allocations can be fully characterized by an interval of consumption levels (or equivalently relative Pareto weights λ) for each income level. These intervals define the set of relative Pareto weights such that both agents are willing to stay in the risk sharing arrangements. Further, each of the end points in the interval is determined by one of the agents being indifferent between paying back or defaulting. Our figures 1 and 7 show that this characterization remains true in an environment with production, for a given level of aggregate capital K. In particular, the intervals that characterize the constrained efficient allocations for a given level of income ϵ can be recovered from the figures as the intervals of λ such that $\lambda' = \lambda$.

Second, the authors also show that an economy will experience perfect risk sharing in the long run if the intersection of these intervals is not the empty set. In our framework, Figure 1 shows that these intervals have an intersection which determines

[17] This value is actually consistent with the estimates of Lustig (2004), who classifies proprietor's income from farms and partnerships as labor income.

the range of relative wealth positions (λ) that are possible in the ergodic set. Obviously, since we have aggregate shocks and capital accumulation, we need to make sure that the intersection of these intervals is non-empty for all capital levels and aggregate shocks in the stationary distribution of aggregate capital. In contrast, the intersection of these intervals is empty for the parametrization of the model displayed in Figure 7, in which case there is no perfect risk sharing in the long run.

Third, Kocherlakota (1996) shows that, in endowment economies, there is a level of patience β above which perfect risk sharing is the long run outcome. This result seems to be also true in our production economy, since perfect risk sharing does not obtain with a relatively low level of patience, while it obtains in the benchmark case. In particular, our results indicate that this threshold level of patience is not too high in production economies.[18]

Figure 8 displays the path for the aggregate capital stock in the stationary distribution and along some (artificial) business cycle simulations[19]. On the second panel of the figure, the aggregate productivity shock alternates between 10 low and 10 high values. At the same time, we draw 1000 independent samples of the idiosyncratic process of the agents for the same time horizon and we average out the results across these independent samples. Both the time series and the "business cycle" figures show that the aggregate capital stock is indeed higher in the competitive equilibrium. Finally, Figure 9 shows how the expected welfare of an agent changes during these artificial business cycles. Note that, by the law of large numbers, this expected welfare can be interpreted as the aggregate (social) welfare in the stationary distribution that arises if we assign equal weights to both types. Strikingly, we see that welfare is higher under the competitive equilibrium throughout the business cycle. This result suggests that, on average, the higher income in this economy due to a higher capital accumulation offsets the welfare loss due to less risk sharing. Of course, since this allocation is not constrained efficient but satisfies the constraints of the planner's problem by construction, agents will suffer welfare losses during the transition towards the higher capital levels that will more than offset the long run gains.

6. Conclusions

This paper studies an economy with capital accumulation and aggregate risk where households are subject to borrowing constraints that do not allow for default. We first show that the borrowing limits that do not allow for default arise as an equilibrium

[18] Here, it is important to note that all the above observations hold for both the competitive equilibrium and the constrained efficient allocations, with the only exception that in the competitive equilibrium allocations we cannot (explicitly) interpret λ as a temporary Pareto weight.

[19] Note that we only plot Figures 8 and 9 for the alternative parameterization, since the benchmark economy exhibits full risk sharing in the stationary distribution and therefore there will be no difference between the competitive equilibrium and the constrained efficient allocations in the long run.

outcome if the intermediaries are allowed to set them. In this sense, we provide further micro foundations for the endogenous borrowing limits.

Moreover, we show that the equilibrium allocations of our economy are not constrained efficient. Despite this, we show that they solve a similar system of equations as the constrained efficient allocation. This characterization identifies a new source of inefficiency that arises in economies with capital accumulation and limited commitment. Moreover, it also considerably simplifies the equilibrium computation.

We also compare numerically the constrained and competitive equilibrium allocations in our economy. First, we find that the calibration plays a crucial role in the determination of the degree of risk sharing in the long run. Whereas the two economies exhibit perfect risk sharing with a standard calibration, the long run allocations are characterized by imperfect risk sharing if agents become more impatient or the weight of capital income in their total income is lower. Second, while the two economies behave qualitatively very similar, capital accumulation is higher in the competitive equilibrium. This result is robust to alternative autarky penalties and different calibrations of the model. Here, we would like to point out that our result is in contrast to the findings in Davila et al. (2005). As shown by the authors capital is under-accumulated in the competitive equilibrium relative to the constrained efficient allocation in a model with exogenous incomplete markets. In their model, however, a higher aggregate capital has the positive effect of helping the consumption-poor agents, who mostly rely on labor income, whereas in our model it has the negative effect of increasing the incentives to default for all agents. Third, we also find that a higher capital accumulation implies that welfare in the long run is higher in the competitive equilibrium in spite of the fact that this allocation is inefficient. This result indicates that less risk sharing can have non-trivial benefits in production economies due to precautionary capital accumulation.

This setup can be used to study more applied questions as well. For example, using a similar setup, Krueger and Perri (2006) study why the rise in earnings inequality was not accompanied by a similar rise in consumption inequality in the last two decades. They solve for the competitive equilibrium allocations with endogenous borrowing constraints. According to our results, there is a scope for governmental intervention in their setting, as the competitive equilibria (and in particular the level of aggregate savings) is not constrained efficient. A set of important questions then arises. How large is the overall welfare loss, whether it is distributed equally across households with different income and wealth levels and whether there is a simple tax policy which would increase aggregate welfare in their environment. These are interesting issues that we leave for further research.

APPENDIX 1: Proofs

Proof of Proposition 1. (i) We first show that there are no profitable deviations from the equilibrium allocation with limits that are tighter or looser then the ones defined by (16). To see this, first notice that tightening the limits will not increase the profits of any intermediary. Further, we now show that no intermediary can make positive profits by loosening the limits, that is, by setting $A_i'(s^t) \leq A_i(s^t) < 0$ for all s^t and any agent $i \in I$. To do this, consider node $\widetilde{s}$ and assume (without a loss of generality) that $A_i'(\widehat{s}) < A_i(\widehat{s})$ for some node $\widehat{s}|\widetilde{s}$ in which the borrowing constraint is binding for type i agents at the level of wealth $A_i(\widehat{s})$. Under the original prices $q(s^{t+1}|\widetilde{s})$, this implies that type i agents would default next period if node $\widehat{s}|\widetilde{s}$ occurs. Since these households would choose $a_i(\widehat{s}) < A_i(\widehat{s}) < 0$ and default if $\widehat{s}$ occurs, it is easy to see that the intermediary would make negative profits. First define $a_i'(s^{t+1}|\widetilde{s})$ as the asset decision of type i households under the new limits and observe that $a_i'(\widehat{s}) < A_i(\widehat{s}) \leq 0$ under $q(\widehat{s}|\widetilde{s})$. Then, default of type i households imply that the profits of the intermediary are given by:

$$
\begin{aligned}
& -k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) + q(\widehat{s}|\widetilde{s})a_i'(\widehat{s}) \\
< \; & -k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) = 0.
\end{aligned}
$$

The second equality follows from the equilibrium condition of the intermediaries in (10). Note that the above equation implies that he cannot break even if he is able to charge a lower price than $q(\widehat{s}|\widetilde{s})$, because type i agents will default in state-date $\widehat{s}$ with certainty. Obviously, he cannot reduce the price for other agents who borrow or increase the price for another agents who save to increase his profits, because those agents will prefer the original prices which are offered by the rest of the intermediation sector.

(ii) We now show that there does not exist any symmetric equilibrium which allows for default. To do this, we assume there exists an equilibrium with prices q and limits $\{A_i\}_{i\in I}$ such that agents of type i would default under some continuation history $s^{t+1}|s^t = \widehat{s}|s^t$ if the current history is $s^t = \widetilde{s}$. First, notice that perfect competition would still require that intermediaries will make zero profits, which would imply that:

$$
-k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) + q(\widehat{s}|\widetilde{s})a_i(\widehat{s}) = 0.
$$

Since household i would only default at node $\widehat{s}$ if $a_i(\widehat{s}) < 0$, the previous equation implies that:

$$
-k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) > 0.
$$

Thus, in any symmetric equilibrium with default, it must be the case that:

$$
\sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)] - 1 > 0.
$$

The previous condition implies that any intermediary could make arbitrarily high positive profits by trading only with agents of type $j \neq i$ and by demanding arbitrary large amounts of total deposits ($\sum_{j \neq i} \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s}) a_j(s^{t+1}|\widetilde{s})$) from them. However, this contradicts the fact that the original portfolio was optimal for the intermediaries under $q(s^{t+1}|s^t)$.

(iii) We now show that there does not exist any symmetric equilibrium with binding limits that are tighter than the endogenous borrowing limits satisfying (16). To do this, we assume there exists an equilibrium with prices q and limits $\{A_i\}_{i \in I}$ such that in state $\widetilde{s}$ the limits are such that $A_i(\widehat{s}|\widetilde{s}) > \tilde{A}_i(\widehat{s}|\widetilde{s})$ where $\tilde{A}_i(\widehat{s}|\widetilde{s})$ would be the limit satisfying (16) and type i agents are borrowing constrained, that is $a_i(\widehat{s}|\widetilde{s}) = A_i(\widehat{s}|\widetilde{s})$ and from (13), we have that

$$q(s^{t+1}|s^t) > \beta\pi(s^{t+1}|s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\}. \tag{32}$$

Notice that perfect competition would still require that intermediaries make zero profits with prices q. In addition, by continuity, (32) implies that there will be a lower price $\tilde{q}(\widehat{s}|\widetilde{s})$ that is close enough to $q(\widehat{s}|\widetilde{s})$ such the type i agents would be willing to borrow more than $a_i(\widehat{s}|\widetilde{s}) = A_i(\widehat{s}|\widetilde{s})$ with this new price $\tilde{q}(\widehat{s}|\widetilde{s}) < q(\widehat{s}|\widetilde{s})$. As long as the intermediary is lending less than $\tilde{A}_i(\widehat{s}|\widetilde{s})$, agent i will not default and the intermediary will increase its profits by this deviation as he can resell these claims for continuation state $\widehat{s}|\widetilde{s}$ for $q(\widehat{s}|\widetilde{s}) > \tilde{q}(\widehat{s}|\widetilde{s})$. Hence, we cannot have a competitive equilibrium with binding limits that are tighter than the endogenous limits defined by (16).■

Proof of Proposition 2. Given the prices $\{Q\}$ in the competitive equilibrium with participation constraints, define the prices in the competitive equilibrium with solvency constraints as follows:

$$q\left(s^{t+1}|s^t\right) = \frac{Q\left(s^{t+1}|s^0\right)}{Q\left(s^t|s^0\right)}$$

Clearly, the factor prices $\{w, r\}$ and the aggregate capital stock $\{K\} = \{k\}$ that satisfy the optimality conditions of the firm (18)-(19) in the competitive equilibrium with participation constraints also satisfy the optimality conditions of the firm (8)-(9) in the competitive equilibrium with solvency constraints.

We now show that the allocations that satisfy the optimality condition of the intermediary (20) in the competitive equilibrium with participation constraints also satisfy the optimality condition of the intermediary (10) in the competitive equilibrium with solvency constraints. This follows from the fact that conditions (10) and (20) in the two equilibria can be written as follows:

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\beta \max_{i \in I} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} F_K\left(s^{t+1}\right). \tag{33}$$

To see that this is the case, consider first the competitive equilibrium with solvency constraints. First, the portfolio constraint in (12) cannot be binding for all agent types. It therefore follows that $\varsigma_i(s^{t+1}) = 0$ for at least one household type and the households' optimality condition in (13) can be rewritten as:

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \max_{i\in I} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\}.$$

Second, substituting for the Arrow price $q(s^{t+1}|s^t)$ and the interest rate $r\left(s^{t+1}\right) = F_K\left(s^{t+1}\right)$ in (10), we obtain (33). Consider now the competitive equilibrium with participation constraints. First, it will be useful to define the following auxiliary multiplier:

$$\upsilon_i(s^t) = \frac{\gamma_i(s^t)}{\mu_i(s^{t-1}) + 1} \text{ for } i \in I.$$

where γ_i is the multiplier on the participation constraint of agent i and μ_i is the recursive co-state variable. Since $\mu_i(s^{t-1}) + 1 > 0$, it follows that $\upsilon_i(s^t) > 0$ only if $\gamma_i(s^t) > 0$. In other words, the multiplier υ_i is positive only when the participation constraint of type $i \in I$ is binding. Second, using the expression for υ_i and the optimality condition for the households in (24), the ratio of Arrow Debreu prices can be written as:

$$\frac{Q\left(s^{t+1}|s^0\right)}{Q\left(s^t|s^0\right)} = \pi(s^{t+1}|s^t)\beta\frac{u'\left(c_i(s^{t+1})\right)\left(1+\upsilon_i\left(s^{t+1}\right)\right)}{u'\left(c_i(s^t)\right)} = \pi(s^{t+1}|s^t)\beta \max_{i\in I}\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}.$$

where we have used the fact that the multiplier on the budget constraint can be set to

$$\eta_i = \frac{\pi(s^0)\beta^0 u'(c_i(s^0))(\mu_i(s^0)+1)}{Q(s^0|s^0)} = u'(c_i(s^0))(\mu_i(s^0)+1)$$

and the last equality follows from the properties of υ_i. If we substitute for the ratio of Arrow Debreu prices and the interest rate $r\left(s^{t+1}\right) = F_K\left(s^{t+1}\right)$ in (20), we also obtain (33), as claimed.

Since the high implied interest rate condition holds, we can then use the prices $\{Q\}$ and the consumption allocations $\{c_i\}_{i\in I}$ from the competitive equilibrium with participation constraints to construct the asset holdings $\{a_i\}_{i\in I}$ that satisfy the budget constraint of the households in the competitive equilibrium with solvency constraints. These are equal to:

$$a_i(s^t) = \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t)\left[c_i(s^{t+n}) - w\left(s^{t+n}\right)\epsilon_i\left(s^{t+n}\right)\right] \tag{34}$$

and

$$a_i(s^0) = \sum_{t=0}^{\infty}\sum_{s^t|s^0} Q(s^t|s^0)\left[c_i(s^t) - w\left(s^t\right)\epsilon_i\left(s^t\right)\right]. \tag{35}$$

Concerning the trading limits, if $\upsilon_i(s^t) = 0$ for agent i in the competitive equilibrium with participation constraints, we first set $A_i(s^{t+1}) = -\sum_{n=1}^{\infty}\sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t)w(s^{t+n})\epsilon_i\left(s^{t+n}\right)$

and we will redefine this limit later. Further, if $\upsilon_i(s^t) > 0$, we set $A_i(s^{t+1}) = a_i(s^{t+1})$, implying that the limit in the competitive equilibrium with solvency constraints will be binding exactly when the participation constraint in (22) is binding.

To make sure that the optimality conditions of the households are satisfied, we can use $q(s^{t+1}|s^t)$ to define the multiplier $\zeta_i(s^{t+1})$ so that the Euler condition in (13) holds. It is easy to check that the multiplier will have the desired properties. In particular, if $\upsilon_i(s^{t+1}) = 0$, $\zeta_i(s^{t+1}) = 0$. Further, if $\upsilon_i(s^{t+1}) > 0$, it follows that $\zeta_i(s^{t+1}) > 0$.

The transversality condition is satisfied, since:

$$
\begin{aligned}
& \lim_{t\to\infty} \sum_{s^t} \beta^t \pi(s^t) u'\left(c_i(s^t)\right) \left[a_i(s^t) - A_i(s^t)\right] \\
\leq\ & \lim_{t\to\infty} \sum_{s^t} \beta^t \pi(s^t) u'\left(c_i(s^t)\right) \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n})\right] \\
\leq\ & u'\left(c_i(s^0)\right) \lim_{t\to\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{u'\left(c_i(s^t)\right)}{u'\left(c_i(s^0)\right)} \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n})\right] \\
\leq\ & u'\left(c_i(s^0)\right) \lim_{t\to\infty} \sum_{s^t} Q(s^t|s^0) \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n})\right] = 0.
\end{aligned}
$$

The first inequality follows from the fact that $[a_i(s^t) - A_i(s^t)]$ is equal to zero if the participation constraint is binding. Further, it is equal to $\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \geq 0$ otherwise, since in this case we have that $a_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \left[c_i(s^{t+n}) - w\left(s^{t+n}\right) \epsilon_i\left(s^{t+n}\right)\right]$ and $A_i(s^{t+1}) = -\sum_{n=1}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) w(s^{t+n}) \epsilon_i\left(s^{t+n}\right)$. The second inequality follows from the fact that $c_i(s^t) \leq \sum_i c_i(s^t)$. The third inequality follows from the the definition of $Q(s^t|s^0)$ and from the fact that $Q(s^t|s^0) \geq \beta^t \pi(s^t) \frac{u'\left(c_i(s^t)\right)}{u'\left(c_i(s^0)\right)}$ by construction. Finally, the last equality follows form the high implied interest rate condition.

Finally, we can construct the value functions $W(a_i(s^t); S_i(s^t))$ and $V(S_i(s^t))$ from the value functions of the competitive equilibrium with participation constraints and redefine the borrowing constraints on Arrow security holdings so that they satisfy $W(A_i(s^{t+1}); S_i(s^{t+1})) = V(S_i(s^{t+1}))$ at every node. Since these limits do not bind for the originally unconstrained consumers, the constructed allocations are still feasible and optimal. ■

Proof of Proposition 3. Given the Arrow prices $\{q\}$ in the competitive equilibrium with solvency constraints, define the prices in the competitive equilibrium with solvency constraints as follows:

$$Q(s^t|s^0) = q(s^t|s^{t-1}) q(s^{t-1}|s^{t-2}) ... q(s^1|s^0).$$

Clearly, the factor prices $\{w, r\}$ that satisfy the optimality conditions of firms in the competitive equilibrium with solvency constraints also satisfy the optimality conditions of the firms in the competitive equilibrium with participation constraints. In addition, using the same arguments as in the proof of proposition 2, it is easy to show that the allocations that satisfy the optimality condition of the intermediary in the competitive equilibrium with solvency constraints also satisfy the optimality condition of the intermediary in the competitive equilibrium with participation constraints, namely

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \max_{i \in I} \left[\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right] F_K\left(s^{t+1}\right) \right\}$$

Substituting forward for $(a_i)_{i \in I}$ in the budget constraint of the competitive equilibrium with solvency constraints and using the expression for $\{Q\}$ defined above, it is easy to see that the consumption allocations $(c_i)_{i \in I}$ that satisfy the budget constraint of the competitive equilibrium with solvency constraint also satisfy the budget constraint of the households in the competitive equilibrium with participation constraints at these prices. Moreover, since the asset holdings $(a_i)_{i \in I}$ in the competitive equilibrium with solvency constraints are subject to portfolio restrictions $\{A_i\}_{i \in I}$ that are not too tight, the value functions $W(a_i(s^t); S_i(s^t))$ and $V(S_i(s^t))$ satisfy the participation constraints in (22).

Note that the allocations in the competitive equilibrium with solvency constraints still solve the same problem if the borrowing constraints on the Arrow securities of the unconstrained households are substituted for the natural borrowing limits defined by:

$$A_i(s^{t+1}) = -\sum_{n=1}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) w\left(s^{t+n}\right) \epsilon_i\left(s^{t+n}\right) \tag{36}$$

and optimality implies that the previous limit is finite.[20] In addition, since the shocks z and ϵ lie in a compact set, the present values of K and f_L are finite, we can use the resource constraint to show that the allocation of the competitive equilibrium with solvency constraints satisfies the high implied interest rate condition.

To make sure that the optimality condition for households in the competitive equilibrium with participation constraints is satisfied, the multipliers $(\mu_i)_{i \in I}$, $(\eta_i)_{i \in I}$, $(\gamma_i)_{i \in I}$ and $(\upsilon_i)_{i \in I}$ can be recovered as follows. First, assume, without loss of generality, that the portfolio constraint in the competitive equilibrium with solvency constraints is not binding for household i at node s^t, hence we set $\upsilon_i(s^t) = 0$ and $\gamma_i\left(s^t\right) = 0$ for $i \in I$. For any other agent type j, $\upsilon_j(s^t)$ is recovered from:

$$\frac{u'\left(c_i(s^t)\right)}{u'\left(c_j(s^t)\right)} = (1 + \upsilon_j(s^t)) \frac{u'\left(c_i(s^{t-1})\right)}{u'\left(c_j(s^{t-1})\right)} \tag{37}$$

[20] In an exchange economy context with sequential trade and potentially incomplete financial markets, Santos and Woodford (1997) show that the natural borrowing limit implied by the optimal allocations has to be finite. Otherwise, one can construct a portfolio that yields more utility than the optimal allocation. The same proof can be used in the present setup.

We can then recover $\gamma_i(s^t)$ and $\mu_i(s^t)$ from the definition of υ_i and from the law of motion of μ_i :

$$\begin{aligned} \upsilon_i(s^t) &= \frac{\gamma_i(s^t)}{1+\mu_i(s^{t-1})} \\ \mu_i(s^t) &= \mu_i(s^{t-1}) + \gamma_i(s^t) \end{aligned}$$

and the multiplier η_i is given by $\eta_i = u'(c_i(s^0))(1+\gamma_i(s^0))$.This guarantees that the consumption allocations of the competitive equilibrium with solvency constraints also satisfy the optimality condition of the households in the competitive equilibrium with participation constraints.■

Proof of Proposition 4. The factor prices $w(s^t)$ and $r(s^t)$ that satisfy the optimality conditions of the firm in the competitive equilibrium can be constructed from the capital levels of the original allocation using equations (8)-(9). Given the consumption allocations $\{c_i\}_{i\in I}$ that solve equations (2), (6) and (29), we can use equations (27) and (26) to define the prices $Q(s^{t+1}|s^t) = Q_p(s^{t+1}|s^t)$ and $q(s^{t+1}|s^t) = q_p(s^{t+1}|s^t)$ that satisfy (13). Since the high implied interest rate condition holds, we can then use these prices and the consumption allocations in order to construct the asset holdings $\{a_i\}_{i\in I}$ that satisfy the budget constraint of the households in the competitive equilibrium (equation (11)) exactly as in the proof of Proposition 2 (see equations (34) and (35)).

Note that the initial value for the ratio of marginal utilities pins down the initial asset levels in the competitive equilibrium. It is also easy to see that an allocation that satisfies condition (29) also satisfies the competitive equilibrium condition of the intermediary in (28).Finally, we can determine the limits $A_i(s^{t+1})$ and show that the transversality condition is satisfied exactly as in the latter part of the proof of Proposition 2. ■

Proof of Proposition 5. To prove the proposition, we first note that the resource constraint in (5) is satisfied by the competitive equilibrium allocations. Since the asset holdings are subject to portfolio restrictions $\{A_i\}_{i\in I}$ that are not too tight, the value functions in the competitive equilibrium satisfy:

$$W^{ce}(a_i(s^t), S_i(s^t)) \geq V^{ce}(S_i(s^t))$$

for all $i \in I$ and all $s^t \in N$, where $W^{ce}(a_i(s^t), S_i(s^t)) = \sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\pi(s^r|s^t)u(c_i(s^r))$ and $V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\pi(s^r|s^t)u(w(s^r)\epsilon_i(s^r))$. Given this, the functions defined by $W(S_i(s^t)) = W^{ce}(a_i(s^t), S_i(s^t))$ and $V(S_i(s^t)) = V^{ce}(S_i(s^t))$ satisfy the participation constraints in (2). Similarly to the proof of Proposition 3, we can show that the competitive equilibrium allocations still solve the same problem if the borrowing constraints on the Arrow securities of the unconstrained households are substituted for the natural borrowing limits (see equation (36)) and that the competitive equilibrium allocation satisfies the high implied interest rate condition.

To recover the multipliers $\{\lambda\}$ and $\{v_i\}_{i\in I}$, we can first use the equilibrium consumption allocations to define $\frac{u'(c_i(s^t))}{u'(c_j(s^t))}$ for any $i, j \in I$. Further, $\{v_i\}_{i\in I}$ can be recovered as in the proof of proposition 3 (see equations (37) and (**??**)). Clearly, this implies that equation (6) is satisfied. In addition, the zero profit condition in the decentralized solution can be rewritten as:

$$\begin{aligned} 1 &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \max_{i\in I} \left[\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right] F_K\left(s^{t+1}\right) \right\} \\ &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \left[\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} (1 + v_i(s^{t+1})) \right] F_K(s^{t+1}) \right\} \end{aligned}$$

Given this, the Euler equation in (29) is also satisfied.■

APPENDIX 2: Computational Method

The implementation of the algorithm used to solve for the two economies is described in what follows. We initially define a grid on the endogenous state space, given by the relative weight λ and by the aggregate capital stock K. Note that the grid on the exogenous state space (ϵ, z) is implicitly defined by our Markov chain assumption. We let $s_1 = [\epsilon, \lambda; z, K]$ and $s_2 = [1-\epsilon, 1/\lambda; z, K]$.

Given the grid on λ, K and (ϵ, z), our procedure finds continuous equilibrium policy functions for the individual consumptions $c_1 \equiv c(\epsilon, \lambda; z, K)$ and $c_2 \equiv c(1-\epsilon, 1/\lambda; z, K)$, the multipliers $v_1 \equiv v(\epsilon, \lambda; z, K)$ and $v_2 \equiv v(1-\epsilon, 1/\lambda; z, K)$, the next period relative weight $\lambda' \equiv \lambda(\epsilon, \lambda, K, z)$, and the law of motion for the aggregate capital stock $K' \equiv K(\epsilon, \lambda, K, z)$, such that all the equilibrium conditions are satisfied.

To find the solution for a given grid point, we start assuming that the participation constraints are not binding for any of the two types. This implies that $v_i = 0$ for $i = 1, 2$ and $\lambda' = \lambda$. Using the equilibrium policy functions, the value functions $W_1 \equiv W(\epsilon, \lambda; z, K)$, $W_2 \equiv W(1-\epsilon, 1/\lambda; z, K)$, $V_1 = V(\epsilon, \lambda, K, z)$ and $V_2 = V(1-\epsilon, 1/\lambda; z, K)$ are calculated recursively and we then check if the participation constraints are actually binding. If they are not binding, the solution is correct. Otherwise, we impose the participation constraint and recalculate the solution.

All the previous objects are approximated with continuous functions using linear interpolation over the finite and endogenous grid, and the procedure is repeated until convergence. More precisely, given a set of functions of interest $h = [\{(c_i, v_i)_{i=1,2}, \lambda', K']$, let T be a non-linear operator such that $T[h\ W\ V]$ satisfies the equilibrium system of equations and the participation constraints. The solution to our problem is then a fixed point of T, i.e., a vector $[h\ W\ V]$ such that $[h\ W\ V] = T[h\ W\ V]$. To approximate the fixed point, we follow the steps below.

Step 1: Guess an initial vector $[h_0 \; W_0 \; V_0]$, where $h_0 = [(c_{i0}, v_{i0})_{i=1,2}, \lambda_0', K_0']$.

Step 2: For each iteration $n \geq 1$, use the previous guess $[h_{n-1} \; W_{n-1} \; V_{n-1}]$ to compute the new vector $[h_n \; W_n \; V_n]$ that satisfies the equilibrium conditions. Here, we have to make sure that the participation constraints are satisfied, that is, $W_n(\epsilon, \lambda_n, K_n, z) \geq V_n(\epsilon, \lambda_n, K_n, z)$.

Step 3: Use $[h_n \; W_n \; V_n]$ as the next initial guess and iterate until $[h_n \; W_n \; V_n]$ converges.

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Appendix 3: Figures

Figure 1: Next Period Wealth Distribution (λ') as a Function of λ and ϵ

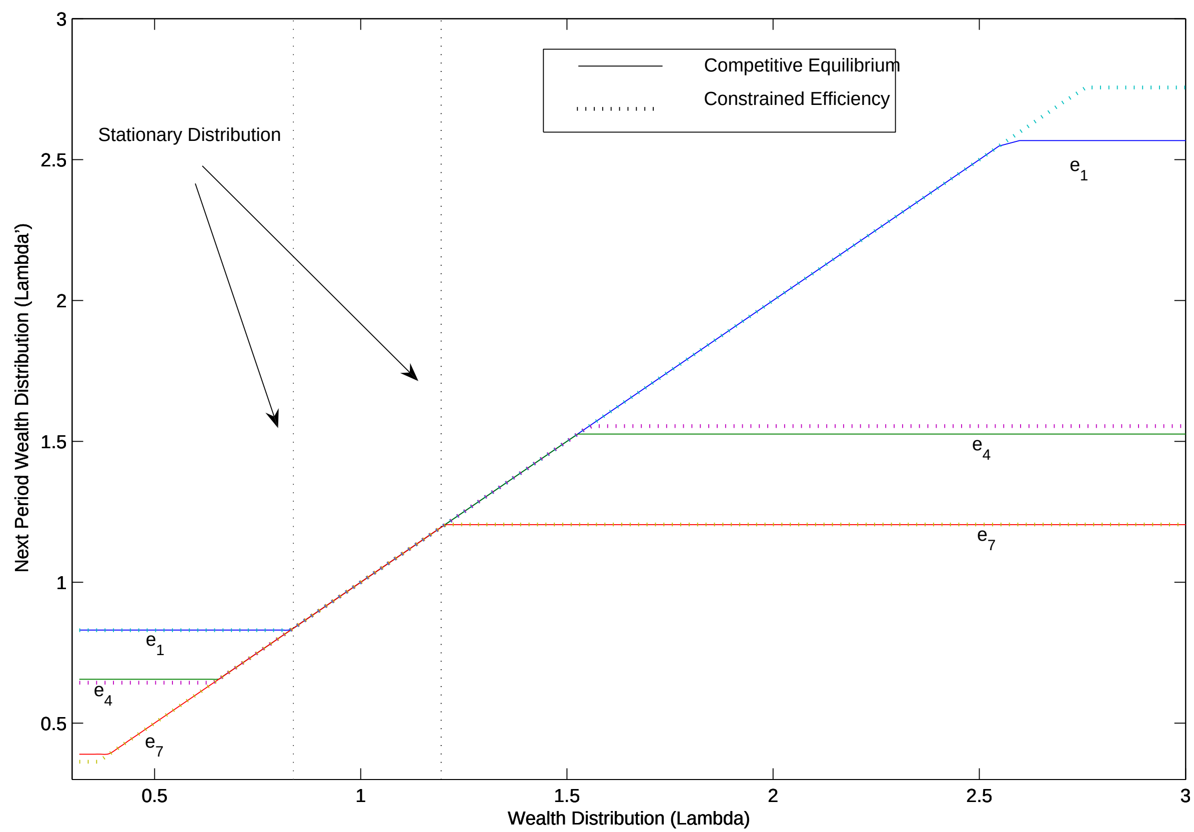


Figure 2: Optimal Consumption (c_1) as a Function of λ and ϵ

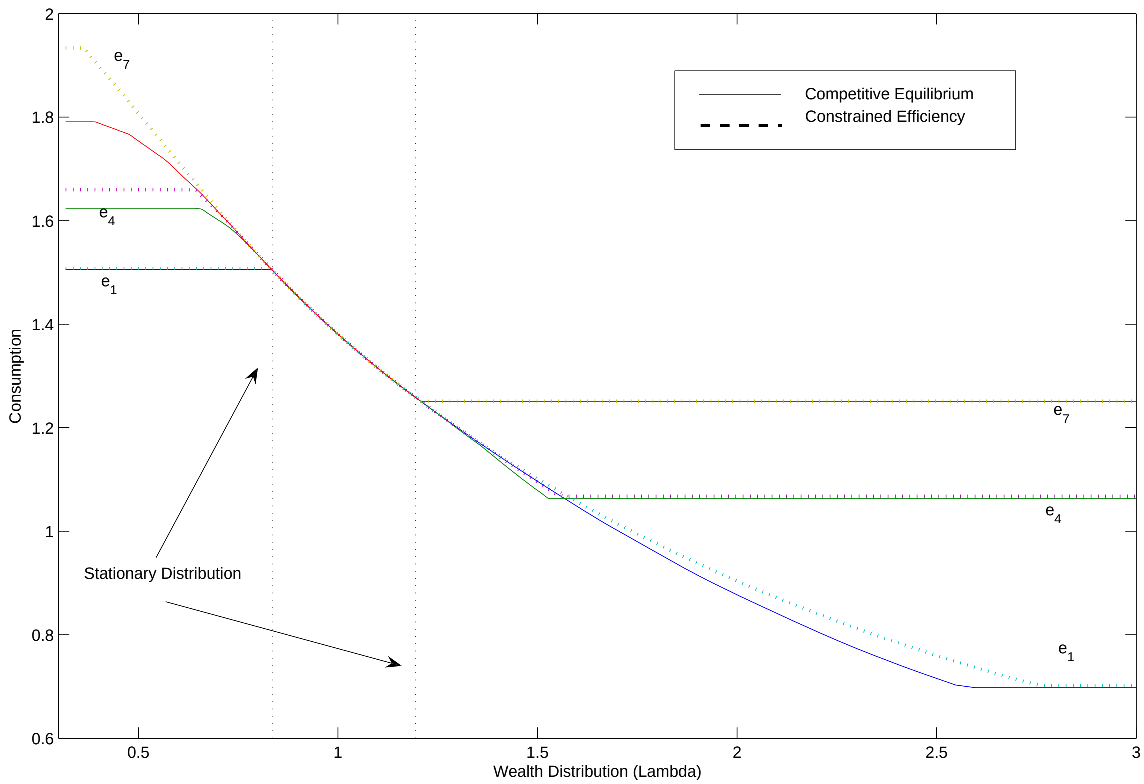


Figure 3: Next Period Capital Stock (K') as a Function of λ and ϵ

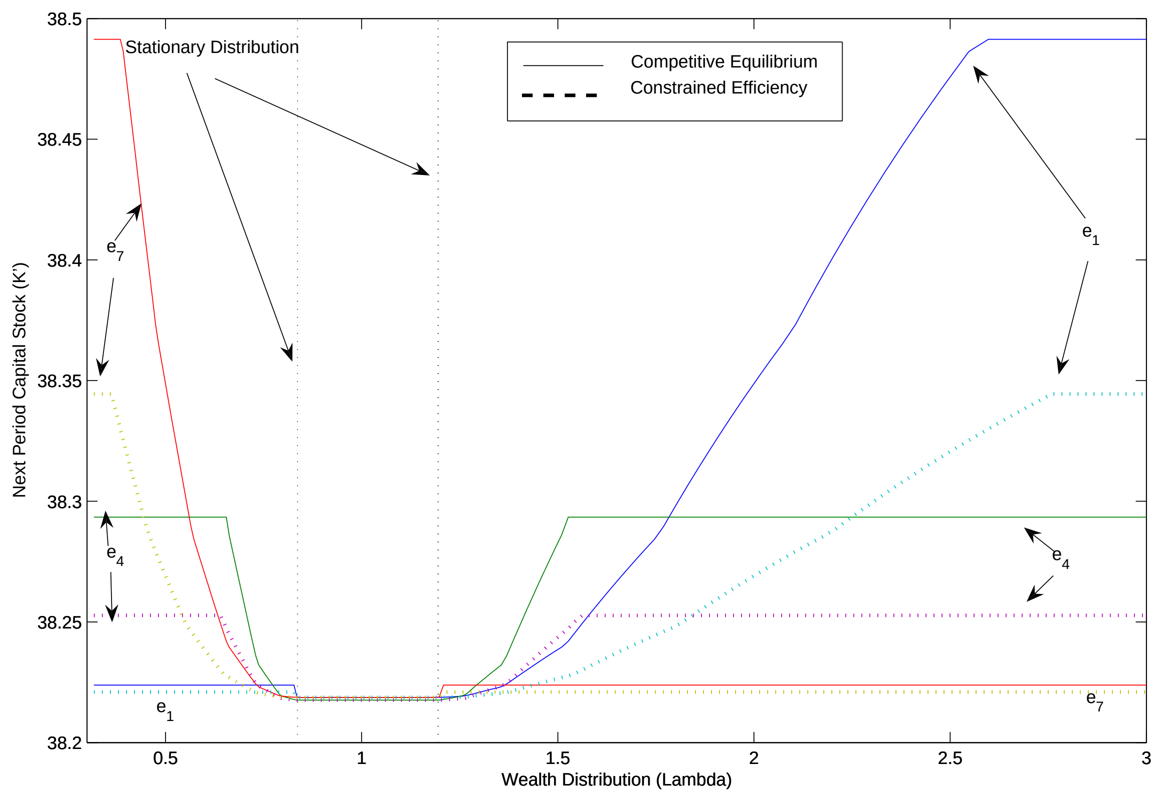


Figure 4: Optimal Consumption (c_1) as a Function of a_1 and ϵ

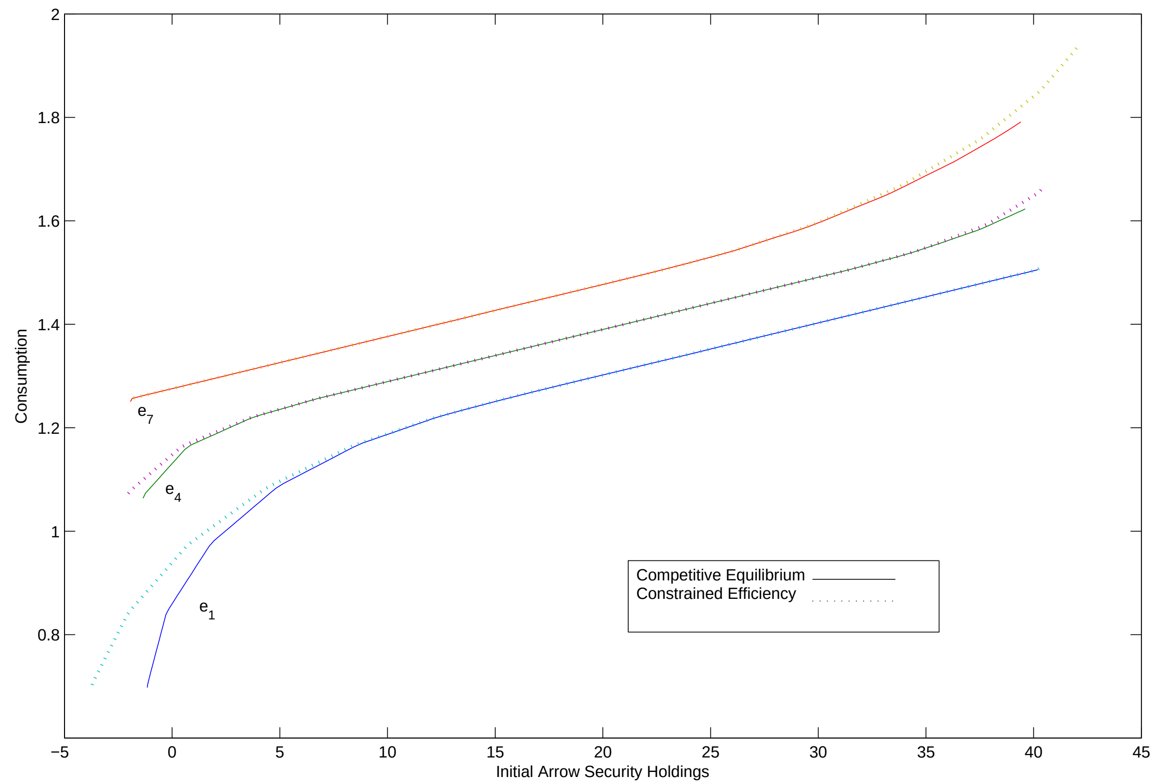


Figure 5: Next Period Capital Stock (K') as a Function of a_1 and ϵ

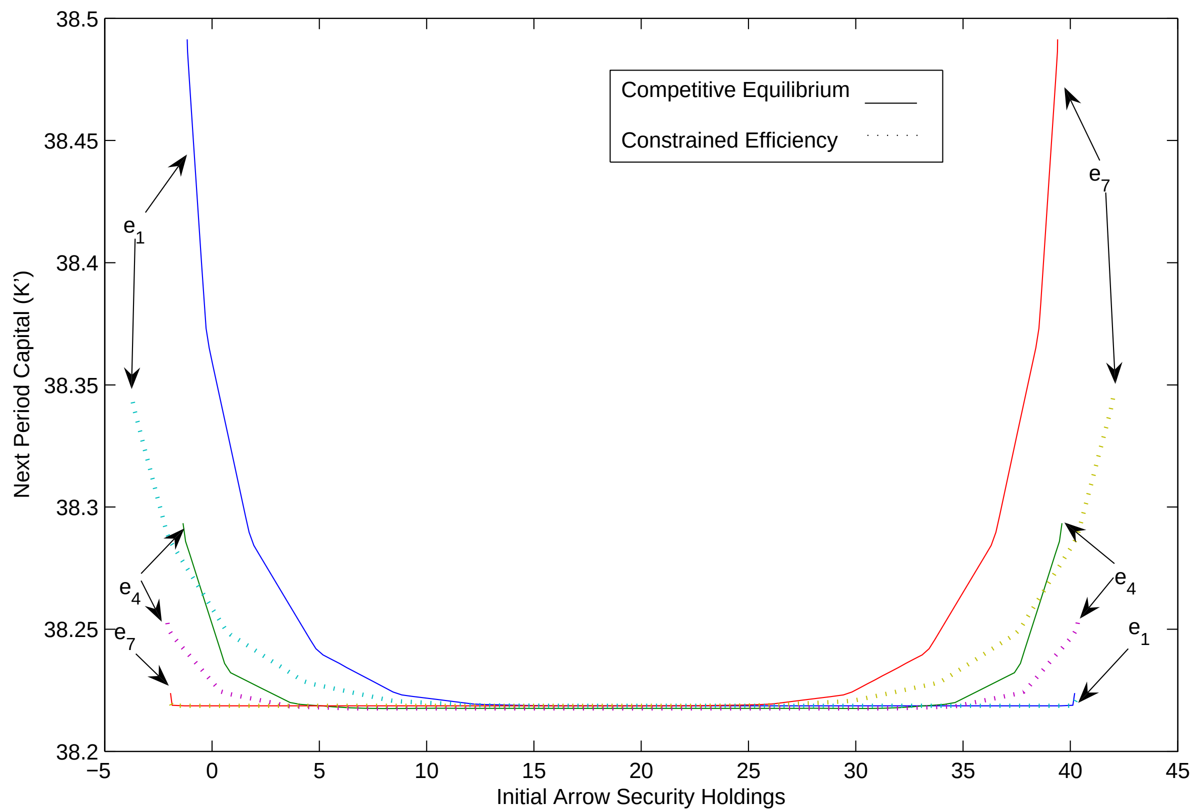
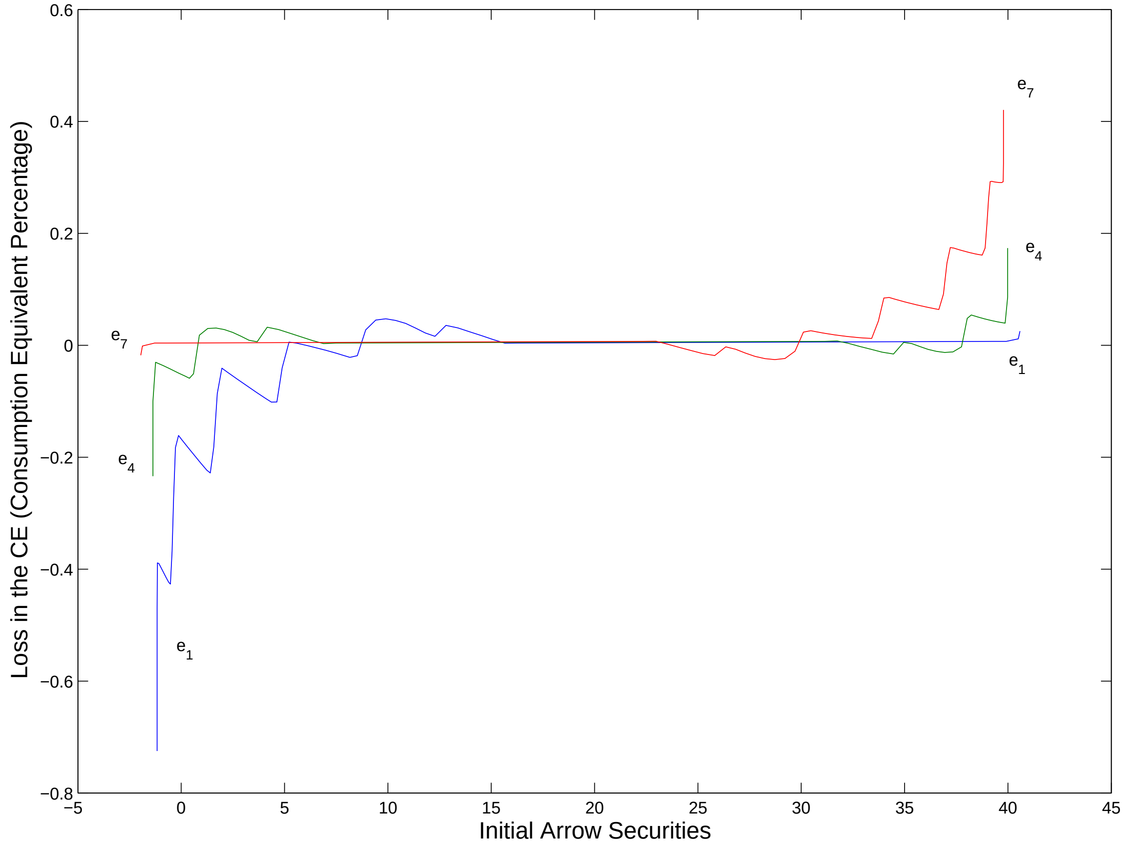


Figure 6: Welfare Loss in the Competitive Equilibrium in Consumption Equivalent (%) Terms as a Function of a_1 and ϵ



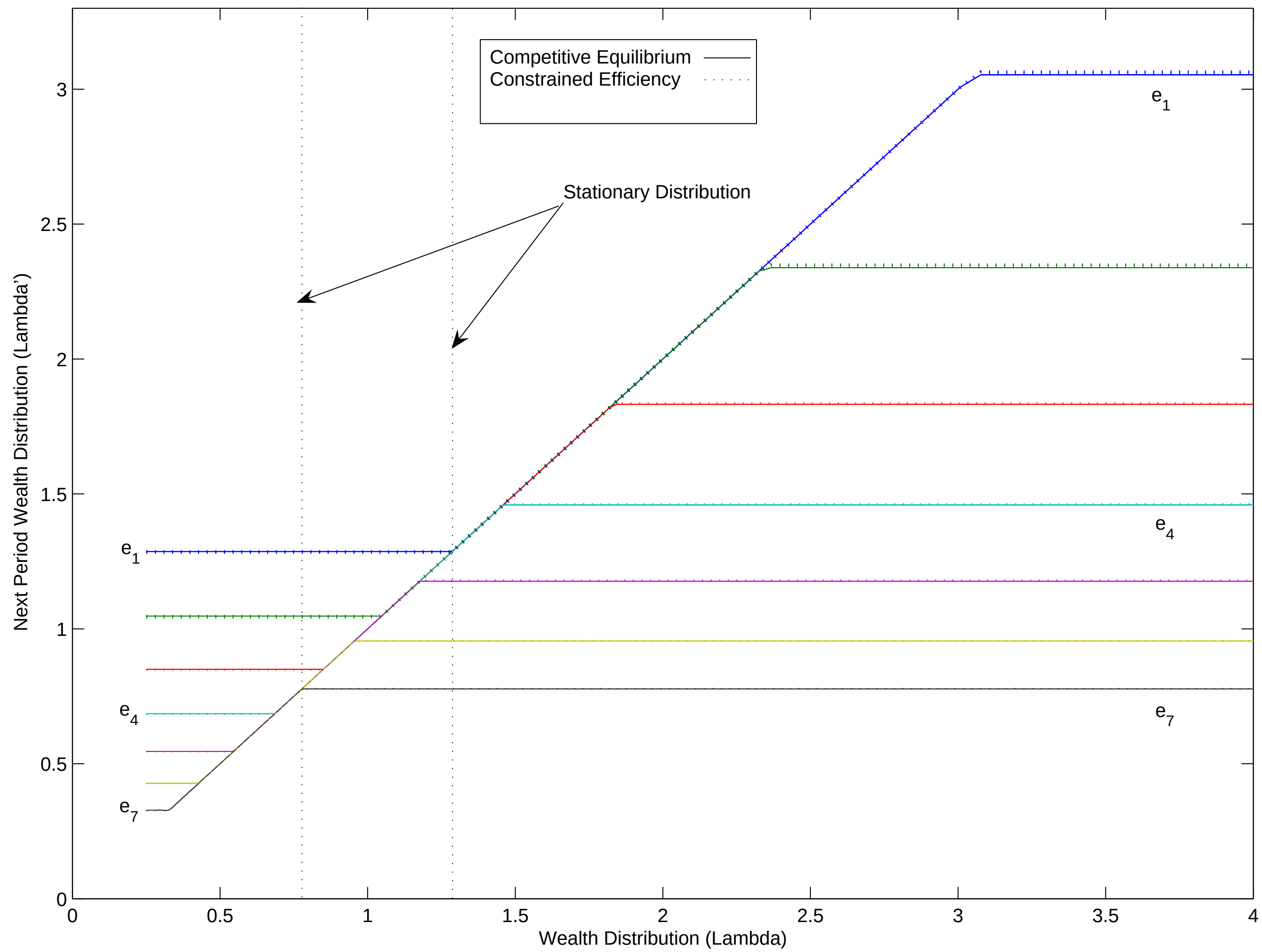

Competitive Equilibrium
Constrained Efficiency
Stationary Distribution
e1
e4
e7
e1
e4
e7
Next Period Wealth Distribution (Lambda')
Wealth Distribution (Lambda)
0
0.5
1
1.5
2
2.5
3
3.5
4

Figure 8: Next Period Capital Stock (K') from Time Series Simulations

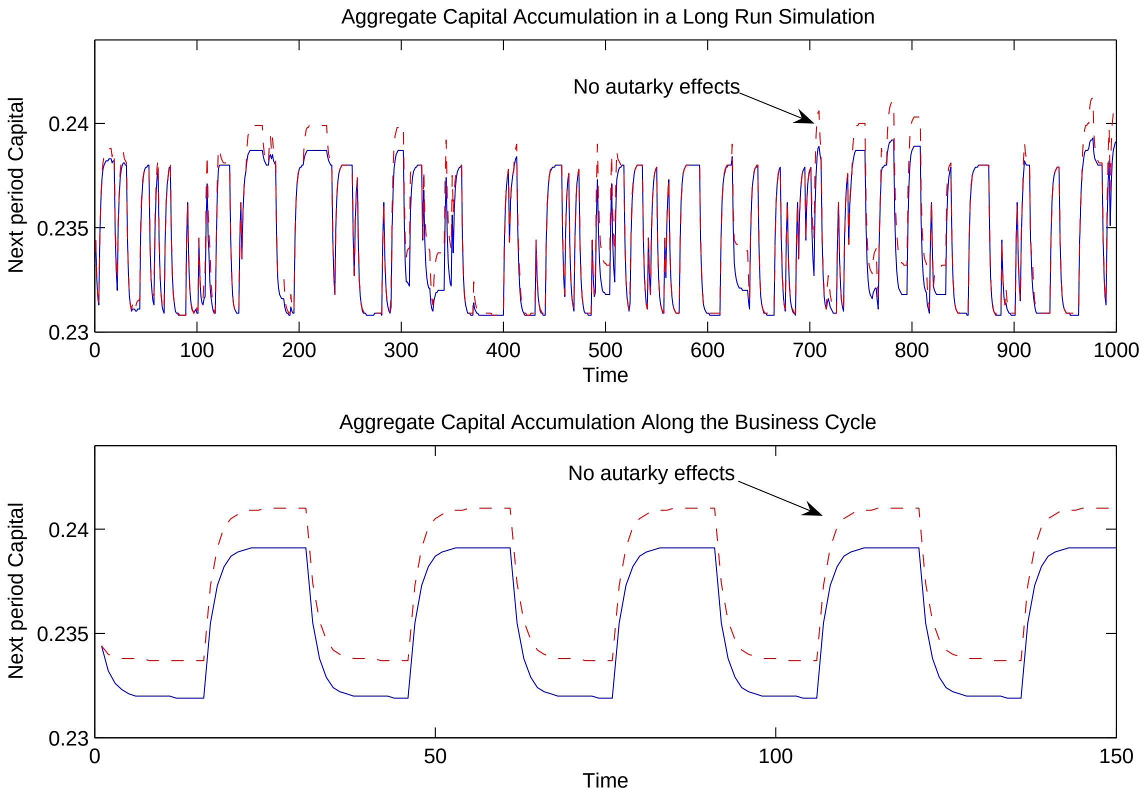


Figure 9: Average Life-Time Utility (W_1) from Time Series Simulations