

## Endogenous incomplete markets, enforcement constraints, and intermediation

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Alvarez and Jermann (2000) show that the constrained efficient allocations of endowment economies with imperfect risk sharing due to limited commitment can be decentralized as competitive equilibria with endogenous debt constraints that are *not too tight*. These are the loosest possible borrowing limits that do not allow for default in equilibrium. However, such a decentralization is not possible in the presence of capital accumulation, since changes in the aggregate capital also affect the incentives to default. In a model with endogenous production, aggregate risk, and competitive intermediaries, we show that a decentralization with endogenous debt constraints is possible if one also imposes an upper limit on the intermediaries' capital holdings.

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### 1. INTRODUCTION

During the recent years, models with limited commitment have been used to analyze important economic issues. Among others, [Thomas and Worrall \(1988\)](#) study efficient wage contracts, [Kocherlakota \(1996\)](#) analyzes optimal risk sharing, and [Alvarez and Jermann \(2001\)](#) study asset returns. Whereas the previous literature focuses mostly on closed endowment economies, several papers have recently incorporated capital accumulation into such a context (see e.g. [Krueger and Perri \(2006\)](#), who study the dynamics of income and consumption inequality, or [Wright \(2001\)](#) and [Bai and Zhang \(2006\)](#), who study international capital flows). Limited commitment economies with both capital accumulation and aggregate uncertainty have received less attention. One exception is the work by [Kehoe and Perri \(2002b\)](#), which uses an open economy model with production to analyze international risk sharing.

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In the present paper, we study the consequences of introducing endogenous production into a closed economy framework with aggregate uncertainty and limited commitment. In particular, we focus on the relationship between constrained efficient allocations and competitive equilibria with endogenous borrowing limits by providing a version of the second fundamental theorem of welfare economics for this type of economy. In our environment, limited commitment arises because agents have the option to default on their financial liabilities every period. In particular, we assume that all their assets are seized in the default period, after which they are excluded from future asset trade (risk sharing) permanently. This implies that they have to rely solely on their labor income, which depends on the aggregate capital stock and on aggregate and idiosyncratic productivity. In other words, the outside option (autarky value) depends on both the exogenous and endogenous states of the economy.

As shown by [Alvarez and Jermann \(2000\)](#), the constrained efficient allocations of exchange economies where full risk sharing is precluded due to limited commitment can be decentralized as competitive equilibria with endogenously incomplete markets where the borrowing constraints are “not too tight.” These are the loosest possible borrowing limits that do not allow for default in equilibrium. We first show that this decentralization is not possible if one introduces capital accumulation and aggregate uncertainty into such a framework. The reason is that, in the presence of binding enforcement constraints, higher capital accumulation has two additional effects on the Euler condition that determines aggregate investment. On the one hand, it increases consumption and output next period, decreasing the incentives to default and raising therefore the benefits of higher aggregate capital. On the other hand, it tightens the enforcement constraints through an increase in the outside option (autarky effect), reducing the benefits of more capital. Since the previous two effects drive a wedge between the marginal rates of substitution and transformation, the optimal allocations cannot be decentralized as competitive equilibria, even in the presence of endogenous debt constraints.

This result has been shown also by [Kehoe and Perri \(2002a, 2004\)](#) for a two-sector model where the agents are interpreted as countries. In addition, these authors argue that the constrained efficient allocations can be decentralized with either capital taxes and government default on foreign loans or with endogenous debt and capital accumulation constraints. However, they focus on the first decentralization and only briefly discuss the latter. In contrast, we study a decentralization with debt constraints, since our agents cannot be interpreted as countries and sovereign default therefore makes no sense. Moreover, one of our key extensions is the introduction of competitive financial intermediaries that operate the investment technology. We first show that by introducing intermediaries we can eliminate one of the two distortions of limited commitment on capital accumulation. In addition, we show that the optimal allocations can be decentralized with endogenous debt constraints and with capital accumulation constraints on the capital holdings of the intermediaries.

Our findings contribute to a growing literature that studies models with limited commitment and capital accumulation. They provide the basic insight regarding why the endogeneity of capital accumulation prevents the decentralization of the constrained

efficient allocations in an environment with endogenous portfolio constraints and limited commitment. In contrast to the model of [Kehoe and Perri \(2004\)](#), who study an open economy where the agents (countries) can keep their stock of capital after default, our model has a very different closed economy setup where the agents' labor income after default depends on the capital stock even though an agent who defaults is excluded from the future ownership of capital. Hence the economic environment in this paper is critically different from previous models in the way that aggregate capital influences the value of default.

We believe that our framework has many potential applications, such as the study of wealth and consumption inequality and the welfare impact of government policies. In particular, our decentralization result indicates that, in these environments, optimality requires not only that borrowing constraints are individual specific, but also that the aggregate capital stock adjusts in order to balance the default incentives. As shown by [Ábrahám and Cárceles-Poveda \(2006\)](#), who study the present setup numerically, the general equilibrium capital accumulation effects that we have identified may play a very important role in these applications.

The paper is organized as follows. [Section 2](#) introduces the model economy. [Section 3](#) discusses the competitive equilibrium with endogenous borrowing limits and financial intermediaries that may be subject to capital accumulation constraints. [Section 4](#) characterizes the constrained efficient allocations of the benchmark economy and [Section 5](#) shows that decentralization as a competitive equilibrium with endogenous borrowing limits is possible only in the presence of accumulation constraints on the capital holdings of the intermediaries. Finally, [Section 6](#) summarizes and concludes.

## 2. THE ECONOMY

We consider an infinite-horizon economy with aggregate uncertainty, idiosyncratic risk, endogenous production, and participation constraints.<sup>1</sup> Time is discrete and indexed by  $t = 0, 1, 2, \dots$ . The resolution of uncertainty is represented by an information structure or event-tree  $N$ . Each node  $s^t \in N$ , summarizing the history until date  $t$ , has a finite number of immediate successors, denoted by  $s^{t+1}|s^t$ . We use the notation  $s^r|s^t$  with  $r \geq t$  to indicate that node  $s^r$  belongs to the sub-tree with root  $s^t$ . With the exception of the unique root node  $s^0$  at  $t = 0$ , each node  $s^t$  has a unique predecessor, denoted by  $s^{t-1}$ . The probability of  $s^t$  as of period 0 is denoted by  $\pi(s^t)$ , with  $\pi(s^0) = 1$ . The conditional probability of  $s^r$  given  $s^t$  is represented by  $\pi(s^r|s^t)$ . For notational convenience, for any variable  $x$  we use  $\{x\} = \{x(s^t)\}_{s^t \in N}$  to denote the entire state-contingent sequence.

At each node  $s^t$ , there exists a spot market for a single consumption good  $y(s^t) \in \mathbb{R}_+$ , which is produced with the following aggregate technology:

$$y(s^t) = f(z(s^t), K(s^{t-1}), L(s^t)).$$

Here  $K(s^{t-1}) \in \mathbb{R}_+$  and  $L(s^t) \in \mathbb{R}_+$  denote the aggregate capital and labor respectively, with  $K(s^{-1}) \in \mathbb{R}_{++}$  given, and  $z(s^t) \in \mathbb{R}_{++}$  is a productivity shock that follows

<sup>1</sup>Our model extends the economies in [Kocherlakota \(1996\)](#) and [Alvarez and Jermann \(2000\)](#) to a context with endogenous production.

a stationary Markov chain with  $N_z$  possible values. Given  $z$ , the production function  $f(z, \cdot, \cdot): \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is assumed to be continuously differentiable on the interior of its domain, strictly increasing, strictly concave in  $K$ , and homogeneous of degree one in the two arguments. Moreover, we assume that  $f_{LK}(z, K, L) > 0$ ,  $\lim_{K \rightarrow 0} f_K(z, K, L) = \infty$ , and  $\lim_{L \rightarrow \infty} f_L(z, K, L) = 0$  for all  $K > 0$  and  $L > 0$ . Capital depreciates at a constant rate  $\delta$  and we define  $F(s^t) = y(s^t) + (1 - \delta)K(s^{t-1})$ .

The economy is populated by two types of households that are indexed by  $i \in \{1, 2\} \equiv I$ , with a continuum of identical consumers within each type.<sup>2</sup> Households have additively separable preferences over sequences of consumption  $\{c_i\}$  of the form

$$U(\{c_i\}) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_i(s^t)),$$

where  $\beta \in (0, 1)$  is the subjective discount factor and  $E_0$  denotes the expectation conditional on information at date  $t = 0$ . The period utility function  $u$  is strictly increasing, strictly concave, unbounded below, and continuously differentiable, with  $\lim_{c \rightarrow 0} u'(c) = \infty$  and  $\lim_{c \rightarrow \infty} u'(c) = 0$ .

At each date-state  $s^t$ , households receive a stochastic labor endowment  $\varepsilon_i(s^t)$  that follows a stationary Markov chain with  $N_\varepsilon$  possible values. Households supply labor inelastically, implying that  $L(s^t) = \sum_{i \in I} \varepsilon_i(s^t)$ . They have a potentially history dependent outside option of  $V_i(s^t)$ . Thus, they are subject to a participation constraint of the form

$$\sum_{r=t}^{\infty} \sum_{s^r | s^t} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V_i(s^t) \quad \forall i \in I \text{ and } \forall s^t.$$

Finally, the resource constraint of the economy at  $s^t$  is given by

$$\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t).$$

### 3. COMPETITIVE EQUILIBRIUM

This section defines a competitive equilibrium with endogenous borrowing limits and a competitive intermediation sector for the framework described in Section 2. To do this, we assume that the economy is populated by a representative firm that operates the production technology and by a risk neutral and competitive financial intermediation sector that operates the investment technology. Since we consider only symmetric equilibria where all intermediaries hold the same portfolio, we focus on the representative intermediary.

In each period  $t$ , after observing the realization of the productivity shock, the representative firm rents labor from the households and physical capital from the intermediary to maximize the period profits

$$\max_{K(s^{t-1}), L(s^t)} f(z(s^t), K(s^{t-1}), L(s^t)) - w(s^t)L(s^t) - r(s^t)K(s^{t-1}).$$

<sup>2</sup>All the results in the paper hold for any arbitrary finite number of types, and the assumption of two types is therefore without loss of generality. We adopt the assumption because it simplifies both the notation and the exposition.

Profit maximization implies that factor prices are given by

$$w(s^t) = f_L(s^t) \equiv f_L(z(s^t), K(s^{t-1}), L(s^t)) \quad \forall s^t \quad (1)$$

and

$$r(s^t) = f_K(s^t) \equiv f_K(z(s^t), K(s^{t-1}), L(s^t)) \quad \forall s^t. \quad (2)$$

The representative intermediary lives for two periods. An intermediary born at node  $s^t$  first decides how much capital  $k(s^t)$  to purchase subject to the capital accumulation constraint  $k(s^t) \leq B(s^t)$ . This (potentially binding) constraint plays a central role when we decentralize the constrained efficient allocations. The capital is rented to the firm, earning a rental revenue of  $r(s^{t+1})k(s^t)$  and a liquidation value of  $(1 - \delta)k(s^t)$  the following period. To finance the capital purchases, the intermediary sells the future consumption goods in the spot market for one period ahead contingent claims, which are traded at price  $q(s^{t+1}|s^t)$ . At  $s^t$ , the intermediary solves

$$\max_{k(s^t)} \left\{ -k(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta] k(s^t) \right\} \quad \text{s.t.} \quad k(s^t) \leq B(s^t). \quad (3)$$

If  $\psi(s^t)$  is the multiplier on the capital accumulation constraint in (3), optimality requires that

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta] - \psi(s^t) \quad \forall s^t. \quad (4)$$

Here, it is important to note that  $1 \leq \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta]$  due to the fact that  $\psi(s^t) \geq 0$ . In other words, if the savings constraint is not binding ( $\psi(s^t) = 0$ ), the intermediary makes zero profits. Otherwise, the nonnegative profits at node  $s^t$  are given by

$$d(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta] k(s^t) - k(s^t) = \psi(s^t) k(s^t). \quad (5)$$

We assume that profits are distributed to the households when they are realized, i.e. during the first period of the intermediary's life-cycle. The period before an intermediary starts its business, households own  $\theta_i^0(s^{t-1})$  shares of it, which they can immediately trade at the price  $p(s^t)$ . This price represents the value of an intermediary that will pay dividends next period. At each  $s^t$ , households can also trade in a complete set of state contingent claims to one-period-ahead consumption. The budget constraint of household  $i \in I$  at  $s^t$  is therefore given by:

$$\bar{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) + p(s^t) \theta_i(s^t) \leq d(s^t) \theta_i(s^{t-1}) + a_i(s^t).$$

In this inequality,  $\bar{c}_i(s^t) = c_i(s^t) - p(s^t) \theta_i^0(s^{t-1}) - w(s^t) \varepsilon_i(s^t)$  represents the individual consumption net of the value of initial shares in the intermediaries and of labor income. In addition,  $a_i(s^{t+1})$  and  $\theta_i(s^t)$  represent the amounts of state contingent claims and shares in the intermediary held by  $i \in I$  at the end of period  $t$ .

Market clearing for the state contingent securities requires that the debt issued by the intermediaries matches the demand of the households, that is,  $\sum_i a_i(s^{t+1}) = [r(s^{t+1}) + (1 - \delta)]K(s^t)$ . Further,  $\theta_i^0(s^{t-1})$  is given for  $i = 1, 2$ , while  $\sum_i \theta_i(s^t) = \sum_i \theta_i^0(s^{t-1}) = 1$ . If we denote by  $\omega_i(s^t) \equiv d(s^t)\theta_i(s^{t-1}) + a_i(s^t)$  the initial asset wealth of the household, its optimization problem at  $s^t$  can be written as

$$\begin{aligned} & \max_{\{c_i, a_i, \theta_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.} \\ & \bar{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) \omega_i(s^{t+1}) \leq \omega_i(s^t) \quad \text{and} \quad \omega_i(s^{t+1}) \geq A_i(s^{t+1}). \end{aligned} \quad (6)$$

As reflected by the second inequality in (6), the individual asset wealth is subject to a borrowing constraint of  $A_i(s^{t+1})$ . The equilibrium determination of these limits is discussed later. If  $\gamma_i(s^{t+1}) \geq 0$  is the multiplier on this constraint, the necessary and sufficient first-order conditions with respect to  $a_i(s^{t+1})$  and  $\theta_i(s^t)$  from the maximization problem of household  $i \in I$  imply that

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} + \frac{\gamma_i(s^{t+1})}{u'(c_i(s^t))} \quad \forall s^{t+1}|s^t \quad (7)$$

and

$$p(s^t) = \beta \sum_{s^{t+1}|s^t} \left\{ \pi(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} d(s^{t+1}) + \frac{\gamma_i(s^{t+1})}{u'(c_i(s^t))} d(s^{t+1}) \right\} \quad \forall s^t.$$

Combining these two first-order conditions yields the pricing equations for the shares of the intermediaries:

$$p(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) d(s^{t+1}) \quad \forall s^t. \quad (8)$$

This equation can also be obtained using no arbitrage arguments. It allows us to rewrite the agent's problem as if the decision variable were the next period wealth  $\omega_i(s^{t+1})$  instead of  $a_i(s^{t+1})$  and  $\theta_i(s^t)$  separately. We use this result below in our definition of a competitive equilibrium. This result implies also that there is a continuum of possible combinations of  $a_i(s^{t+1})$  and  $\theta_i(s^t)$  that yield the same allocations, since the share in the intermediaries is a "redundant" asset in spite of markets being endogenously incomplete. Finally, the transversality condition in terms of wealth is given by

$$\lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) [\omega_i(s^t) - A_i(s^t)] \leq 0 \quad \forall s^t.$$

**DEFINITION 1.** A *competitive equilibrium* with borrowing constraints  $\{A_i\}_{i \in I}$ , capital accumulation constraints  $\{B\}$ , and initial conditions  $K(s^{-1})$  and  $\{\omega_i(s^0)\}_{i \in I}$  is a vector of quantities  $\{(c_i, \omega_i)_{i \in I}, k, K, d\}$  and prices  $\{w, r, q\}$  such that (i) given prices,  $\{c_i, \omega_i\}$  solves the problem in (6) for each household  $i \in I$ ; (ii) the factor prices  $\{w, r\}$  satisfy the optimality conditions of the firm in (1) and (2); (iii)  $q, r$ , and  $d$  satisfy the optimality conditions of the intermediary in (4) and (5); and (iv) all markets clear, i.e., for all  $s^t \in N$ ,  $k(s^t) = K(s^t)$ ,  $\sum_i \omega_i(s^{t+1}) = [r(s^{t+1}) + 1 - \delta]K(s^t) + d(s^t)$ ,  $\sum_i \varepsilon_i(s^t) = L(s^t)$ , and  $\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t)$ .

As stated in [Section 2](#), each household  $i$  has an outside option of  $V_i(s^t)$ . In the present setting, we assume that households can leave the risk sharing arrangement at any date-state to go to financial autarky. In this case, they are able only to consume their labor income, while they are excluded from financial markets forever.<sup>3</sup> Given this, we choose limits that are *not too tight* in the sense that looser limits would imply that an agent with that level of debt prefers to leave the trading arrangement. To determine these limits, we define the value of the trading arrangement recursively. The state vector of household  $i \in I$  is represented by  $S_i(s^t) = (\{\varepsilon_i(s^t)\}_{i \in I}, z(s^t), K(s^{t-1}))$ , where  $(\{\varepsilon_i(s^t)\}_{i \in I}, z(s^t))$  is the vector of exogenous states and  $K(s^{t-1})$  is an endogenous state that is determined in equilibrium. Using this notation, the value of the trading arrangement at  $s^t$  can be written as

$$W^{ce}(\omega_i(s^t), S_i(s^t)) = u(c_i(s^t)) + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t) W^{ce}(\omega_i(s^{t+1}), S_i(s^{t+1})).$$

**DEFINITION 2.** The borrowing constraints  $\{A_i\}_{i \in I}$  are *not too tight* if they satisfy the following condition for all  $i \in I$  and all nodes  $s^t \in N$ :

$$W^{ce}(A_i(s^t), S_i(s^t)) = V^{ce}(S_i(s^t)), \quad (9)$$

where the value of the outside option at  $s^t$  is given by

$$V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi(s^r|s^t) u(w(s^r)\varepsilon_i(s^r)).$$

It is important to note that the value of staying in the trading arrangement  $W^{ce}$  is strictly increasing in wealth, whereas the autarky value  $V^{ce}$  is not a function of  $\omega_i(s^t)$ . This implies that the limits defined by (9) exist and are unique under our assumptions on the utility function. Moreover, since  $W^{ce}(0, S_i(s^t)) \geq V^{ce}(S_i(s^t))$  and  $W^{ce}$  is increasing in  $\omega_i$ , equation (9) implies that  $A_i(s^t) \leq 0$ . Intuitively, no agent would default with a positive level of wealth, since he could then afford a higher current consumption than in autarky and would have at least as high a lifetime utility as in autarky from next period on.

Finally, it should be pointed out that all the results in the paper hold for an alternative setting where the intermediaries maximize their market values but are infinitely lived. In this case, each intermediary decides how much capital  $k(s^t)$  to purchase at each node subject to the capital accumulation constraint in (3). The capital is rented to the firm and is financed by selling the next period consumption goods in the spot market for one-period-ahead contingent claims. If we let

$$Q(s^{t+j}|s^t) = q(s^{t+j}|s^{t+j-1})q(s^{t+j-1}|s^{t+j-1}) \dots q(s^{t+1}|s^t)$$

<sup>3</sup>One could consider a different outside option where households are excluded from trade in Arrow securities but can still save by accumulating physical capital. This would not affect our results qualitatively.

be the state  $s^t$  price of consumption delivered at state  $s^{t+1}$ , the intermediary at  $s^t$  solves the problem

$$\max_{\{k\}} \left\{ \sum_{j=0}^{\infty} \sum_{s^{t+j}} Q(s^{t+j}|s^t) ([r(s^{t+j}) + (1 - \delta)]k(s^{t+j-1}) - k(s^{t+j})) \right\}.$$

It is easy to see that equations (4) and (5) still hold in this alternative setting. Further, we can assume that profits are distributed to the households every period according to their beginning-of-period ownership shares  $\theta_i(s^{t-1})$ , where  $\theta_i(s^{-1})$  if given for  $i = 1, 2$ . If we let  $\omega_i(s^t) \equiv [d(s^t) + p(s^t)]\theta_i(s^{t-1}) + a_i(s^t)$  and  $\bar{c}_i(s^t) = c_i(s^t) - w(s^t)\varepsilon_i(s^t)$ , the price of the shares and the budget constraint of the households at  $s^t$  are given by

$$p(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[d(s^{t+1}) + p(s^{t+1})]$$

and

$$\bar{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i(s^{t+1}) + p(s^t)\theta_i(s^t) \leq [d(s^t) + p(s^t)]\theta_i(s^{t-1}) + a_i(s^t).$$

While this alternative setting might be more appealing, since it requires only setting  $\theta_i(s^{-1})$ , it might lead to the typical shareholder disagreement problem under incomplete markets.<sup>4</sup> In other words, when the borrowing constraints are binding, different household types typically value future output differently due to the fact that their marginal rates of substitution are not equalized. Note that this is not an issue if the intermediary lives for two periods, in which case a household who holds the majority of shares at  $s^t$  will agree to use  $q(s^{t+1}|s^t)$  as a discount factor. However, a currently unconstrained agent may prefer a different discount factor if the intermediary is infinitely lived, since she may get constrained in some future contingency. Given this, we chose to work with the two period formulation.

#### 4. CONSTRAINED EFFICIENT ALLOCATIONS

This section characterizes the constrained efficient allocations of the economy in [Section 2](#). As usual, the optimal allocations solve a central planning problem where the planner takes into account both the resource constraint and the participation constraints of the two households. If  $\alpha_i$  is the initial Pareto weight assigned by the planner

<sup>4</sup>For a good review of this literature see [Grossmann and Stiglitz \(1977, 1980\)](#) or [Drèze \(1985\)](#). See also [Duffie and Shafer \(1986\)](#) for a detailed analysis of the value maximization objective in a multiperiod context and [Cárceles-Poveda and Coen-Pirani \(2005, 2006\)](#) for a more formal discussion of this issue in a model where markets are exogenously incomplete.



to each household  $i$ , the problem of the planner at  $s^0$  can be written as follows:

$$\max_{\{c_i\}_{i \in I}, K} \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.} \\ \sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t) \quad \forall s^t \quad (10)$$

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V(S_i(s^t)) \quad \forall i \in I \text{ and } \forall s^t. \quad (11)$$

Note that we have set  $V_i(s^t) = V(S_i(s^t))$  by assuming that the outside option value for  $i \in I$  depends on  $S_i(s^t) = (\{\varepsilon_i(s^t)\}_{i \in I}, z(s^t), K(s^{t-1}))$ .

In the literature, a standard approach to write this problem recursively is to use the agent's lifetime utility as a state variable (see e.g. [Atkeson and Lucas 1995](#), [Thomas and Worrall 1988](#), and [Kocherlakota 1996](#)). To do this, we let  $\varphi(s^t) = (\{\varepsilon_i(s^t)\}_{i \in I}, z(s^t))$  be the vector of exogenous shocks,  $h(s^t) = \{h_i(s^t)\}_{i \in I}$  be the vector of lifetime utilities at  $s^t$ , and  $h(s^{t+1}|s^t) = \{h_i(s^{t+1}|s^t)\}_{i \in I}$  be the vector of next period promised lifetime utilities if date-event  $s^{t+1}$  occurs. Using this notation, the recursive formulation of the problem above is given in (12) to (15):

$$J(h(s^t), \varphi(s^t), K(s^{t-1})) = \max_{\{c_i(s^t)\}_{i \in I}, \{h(s^{t+1}|s^t)\}_{i \in I}, \{K(s^t)\}} \left\{ F(s^t) - K(s^t) - \sum_{i \in I} c_i(s^t) \right. \\ \left. + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t) J(h(s^{t+1}|s^t), \varphi(s^{t+1}), K(s^t)) \right\} \quad (12)$$

s.t.

$$h_i(s^{t+1}|s^t) \geq V(S_i(s^{t+1})) \quad \forall i \in I \text{ and } s^{t+1} \quad (13)$$

$$u(c_i(s^t)) + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t) h_i(s^{t+1}|s^t) \geq h_i(s^t) \quad \forall i \in I \quad (14)$$

$$\sum_{i \in I} c_i(s^t) + K(s^t) - F(s^t) \leq 0. \quad (15)$$

For convenience we have written the problem as that of a component planner whose objective is to minimize the cost (or maximize the surplus) of delivering certain lifetime utilities  $h(s^t)$  to the households.<sup>5</sup> The constraints in (13) represent the participation constraints, while those in (14) make sure that the lifetime utility promises are honored at every date-state  $s^t$ . Finally, condition (15) guarantees that the planner cannot store goods between periods. It is important to note that we need to set  $h(s^0)$  in such a way that  $J(h(s^0), \varphi(s^0), K(s^{-1})) = 0$  to get a solution on the (constrained) Pareto frontier. The lifetime utilities  $h_i(s^0)$  and  $h_{-i}(s^0)$  correspond to the Pareto weights  $\alpha_i$  and  $\alpha_{-i}$  in the sequential problem.

<sup>5</sup>Alternatively, we can use the recursive saddle point method of [Marcet and Marimon \(1998\)](#), as in [Kehoe and Perri \(2002a, 2004\)](#). In this case, we obtain the same first-order conditions.

Several points are worth noting. First, in the absence of the participation constraints in (13), this is a standard convex optimization problem, since we have a concave objective function and a convex set of constraints. However, in the more interesting case where the participation constraints are (occasionally) binding, convexity is not always guaranteed. In particular, a sufficient (but not necessary) condition for convexity is that the autarky value  $V$  is a convex function of the aggregate capital. Since we assume that agents are allowed to keep their labor income in autarky, this value at  $s^t$  can be defined recursively as follows:

$$V(S_i(s^t)) = u(\varepsilon_i(s^t)f_L(z(s^t), K(s^{t-1}), L(s^t))) + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t)V(S_i(s^{t+1})). \quad (16)$$

As we see, the convexity of  $V$  with respect to the aggregate capital stock depends on the curvature of the period utility  $u$ , on the convexity of the marginal product of labor  $f_L$  with respect to capital, and on the optimal law of motion of capital  $K'(\{\varepsilon_i(s^t)\}_{i \in I}, K, z)$ . Since the latter is an endogenous object, however, it is practically impossible to establish analytical conditions under which  $V$  is convex in  $K$ .

In addition, the authors of a few models with capital accumulation and limited commitment have derived sufficient conditions for the convexity of similar problems. However, these models are simpler in several important dimensions and we therefore cannot apply their results. First, [Albuquerque and Hopenhayn \(2004\)](#) derive sufficient conditions when both parties are risk neutral. In this case, it is easy to see that our problem is convex. Second, [Sigouin \(2003\)](#) studies a model where the borrower is risk averse, but where his payoff after default does not depend on the capital stock. Under this assumption also, it can be shown that the first-order conditions are necessary and sufficient. In particular, this is the case with a linear technology, since the marginal product of labor does not depend on the aggregate capital. Unfortunately, neither linear utility nor linear technology are appealing in our framework. Finally, [Thomas and Worrall \(1994\)](#) provide sufficient conditions for a case where the borrower is risk averse and the outside option depends on the aggregate capital. However, capital influences only the value of autarky in the period of default, while the autarky value in our case depends also on the future levels of capital (through the future wages). Thus, the condition they derive is not applicable to our setup either.

The previous discussion illustrates that it is hard to derive sufficient conditions under which the constraint set is convex in the present framework. Given this, we use the following conjecture throughout the text.

**CONJECTURE 1.** *The first-order conditions of the component planner's problem with  $V$  given by (16) are not only necessary but also sufficient.*

We think that this conjecture is not problematic for two reasons. First, one can always introduce lotteries to convexify the constraint set (see e.g. [Ligon et al. 2000](#)), in which case the value function is concave. While the key first-order conditions that we use to derive our main results are essentially the same with randomization, the notation is considerably more complicated. Second, [Ábrahám and Cárceles-Poveda \(2006\)](#)

numerically solve different parameterizations of the problem of finding the constrained efficient allocations with a version of a policy iteration algorithm that imposes all the necessary first-order conditions, including the participation constraints and the Kuhn–Tucker slackness conditions. In all the cases considered, the authors find a unique constrained efficient allocation regardless of the initial guess. Since the first-order conditions of this problem are necessary, this indicates that (at least for the assumed functional forms and for a wide range of parameter values), the first-order conditions are also sufficient.

In what follows, we let  $\mu_i(s^t) \geq 0$  and  $\beta\pi(s^{t+1}|s^t)\gamma_i(s^{t+1}|s^t) \geq 0$  be the Lagrange multipliers of the constraints in (14) and (13) respectively. Under the assumption that the first-order conditions are sufficient, it is easy to see that the optimal allocations of this problem can be characterized by the resource constraint, the participation constraints in (10) and (11), and the following first-order conditions at each date-state:<sup>6</sup>

$$\frac{u'(c_i(s^t))}{u'(c_{-i}(s^t))} = \lambda(s^t) = \frac{1 + v_{-i}(s^t)}{1 + v_i(s^t)} \lambda(s^{t-1}) \quad (17)$$

and

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} (1 + v_i(s^{t+1})) F_K(s^{t+1}) \right\} \\ - \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{j=1,2} \frac{v_j(s^{t+1})}{u'(c_j(s^t))} V_K(S_j(s^{t+1})) \right\} \quad \forall i \in I. \quad (18)$$

The terms  $F_K(s^t) = f_K(s^t) + 1 - \delta$  and  $\{V_K(S_i(s^{t+1}))\}_{i \in I}$  on the right-hand side of (18) represent the derivatives of total output  $F$  and of the outside option value  $V$  with respect to the aggregate capital stock  $K$ . We have expressed the first-order conditions in terms of the normalized multipliers  $\lambda$  and  $v_i$ , which simplify the system of equilibrium equations and are given by

$$v_i(s^t) = \frac{\gamma_i(s^t)}{\mu_i(s^{t-1})} \quad \forall i \in I \text{ and } \forall s^t, t \geq 1$$

and

$$\lambda(s^t) = \frac{\mu_{-i}(s^t)}{\mu_i(s^t)} \quad \forall s^t, t \geq 0.$$

Several points are worth noting. First, since  $\mu_i(s^0) > 0$  due to the fact that the initial constraint in (14) is always binding at the optimum, it follows that  $v_i(s^t) > 0$  only if  $\gamma_i(s^t) > 0$  for all  $s^t$ . This implies that  $v_i(s^t)$  is positive only when the participation constraint of type  $i \in I$  is binding. Second,  $\lambda$  represents the time varying Pareto weight of type 2 households relative to type 1 households. Thus, as usual in models with endogenously incomplete markets, condition (17) implies that the relative consumptions of the

<sup>6</sup>We assume that  $J(h(s^0), \varphi(s^0), K(s^{-1})) = 0$ , implying that the constraint (15) has to be satisfied with equality at all date-states, and that we can replace it by (10). Moreover the definition of  $h$  obviously implies that (13) is equivalent to (11).

two types are determined by their time varying relative Pareto weights. Third, as in other models without commitment (see e.g. [Thomas and Worrall 1988](#) and [Kocherlakota 1996](#)) whenever households of type 1 have a binding participation constraint ( $v_1(s^t) > 0$ ),  $\lambda$  decreases, and their relative Pareto weight therefore increases. The opposite happens when the participation constraint of type 2 household is binding. Finally, since the aggregate technology and the idiosyncratic income shocks are Markovian, the optimal allocation of this problem is recursive in  $(\{\varepsilon_i(s^t)\}_{i \in I}, z, K, \lambda)$ , where  $(\{\varepsilon_i(s^t)\}_{i \in I}, z)$  is the vector of exogenous states and  $(K, \lambda)$  are endogenous states that are determined at the optimum.

When the participation constraints are not binding for any household at any continuation history  $s^{t+1}|s^t$ , implying that  $v_i(s^{t+1}) = 0$  for  $i = 1, 2$ , the Euler equation (18) reduces to the standard capital Euler condition of the stochastic growth model. The presence of binding enforcement constraints at  $s^{t+1}$  introduces two additional effects on the intertemporal allocation of consumption and capital.

First, it increases the planner's marginal rate of substitution between period  $t$  and  $t+1$  goods, raising the benefits of higher aggregate capital at  $t+1$ , since this increases future consumption and decreases the default incentives. This is reflected by the presence of  $v_i(s^{t+1})$  in the first part of the right-hand side of the equation. Second, it tightens the enforcement constraints through an increase in the autarky value, reducing the benefits of more capital at  $t+1$ . This is reflected by the autarky effects in the second part of the right-hand side of the equation.

The following property proves to be useful later on. If  $\{c_i(s^t)\}_{i \in I}$  is constrained efficient and  $W(S_j(s^t)) > V(S_j(s^t))$ , it has to be the case that, at each node,

$$\frac{u'(c_i(s^t))}{u'(c_i(s^{t-1}))} = \max_{j=1,2} \frac{u'(c_j(s^t))}{u'(c_j(s^{t-1}))}.$$

Essentially, this states that unconstrained agents have the maximal marginal rate of substitution in the constrained efficient equilibrium. Note that this can be easily checked if we rewrite equation (17) as

$$\frac{u'(c_i(s^t))}{u'(c_{-i}(s^t))} = \lambda(s^t) = \frac{(1 + v_{-i}(s^t))}{(1 + v_i(s^t))} \frac{u'(c_i(s^{t-1}))}{u'(c_{-i}(s^{t-1}))}.$$

In addition, it implies that, for all  $s^t$ ,

$$\max_{j=1,2} \frac{u'(c_j(s^t))}{u'(c_j(s^{t-1}))} = \frac{u'(c_i(s^t))(1 + v_i(s^t))}{u'(c_i(s^{t-1}))} = \frac{u'(c_{-i}(s^t))(1 + v_{-i}(s^t))}{u'(c_{-i}(s^{t-1}))}. \quad (19)$$

In what follows, we focus on allocations that have high implied interest rates, in the sense that their present value is finite when discounted with the appropriate present value prices.<sup>7</sup> We say that an allocation  $\{c\} \equiv \{c_i + c_{-i}\}$  has high implied interest rates if

$$\sum_{t=0}^{\infty} \sum_{s^t} Q_p(s^t|s^0) c(s^t) < \infty,$$

<sup>7</sup>This assumption is not very restrictive in the present setting, since it is satisfied whenever consumption is bounded away from zero.

where

$$q_p(s^{t+1}|s^t) = \max_{i=1,2} \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} \quad (20)$$

and

$$Q_p(s^t|s^0) = q_p(s^t|s^{t-1})q_p(s^{t-1}|s^{t-2}) \cdots q_p(s^1|s^0). \quad (21)$$

## 5. DECENTRALIZATION WITH CAPITAL ACCUMULATION CONSTRAINTS

This section shows that decentralization of the constrained efficient allocations with debt constraints is possible only if one imposes the savings constraint on the capital holdings of the intermediary. In a related two-country model, [Kehoe and Perri \(2004\)](#) show that decentralization is possible with government default on foreign loans and with capital income taxes. In our closed economy framework, it is less appealing to model governments defaulting on behalf of some of the households against other households in the same economy. Given this, we focus on decentralization with borrowing constraints. As we show below, such decentralization requires that we also impose capital accumulation constraints on the capital holdings of the intermediaries.<sup>8</sup>

We start by showing that constrained efficient allocations with an outside option of financial autarky cannot be decentralized as competitive equilibria in the presence of binding capital accumulation constraints.

**PROPOSITION 1.** *Let  $(\{c_i(s^t)\}_{i \in I}, K)$  be a constrained efficient allocation where  $V_K \neq 0$ . This allocation cannot be decentralized as a competitive equilibrium with only borrowing constraints that are not too tight unless the participation constraints in the constrained efficient allocation never bind.*

**PROOF.** To prove the proposition, we can use equation (19) to rewrite the Euler condition of the planner in (18) as follows:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \max_{j=1,2} \left\{ \frac{u'(c_j(s^{t+1}))}{u'(c_j(s^t))} \right\} F_K(s^{t+1}) - \sum_{j=1,2} \frac{v_j(s^{t+1})}{u'(c_j(s^t))} V_K(S_j(s^{t+1})). \quad (22)$$

Consider now the case where there are no capital accumulation constraints. In this case, the intermediaries always make zero profits, implying that  $d(s^t) = 0$  and  $p(s^t) = 0$  for all  $s^t \in N$ . Hence, households trade only in Arrow securities subject to the following budget and portfolio constraints:

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) \leq a_i(s^t) + w_i(s^t)$$

and

$$a_i(s^{t+1}) \geq A_i(s^{t+1}). \quad (23)$$

<sup>8</sup>These capital accumulation constraints can also be reinterpreted as the equilibrium consequence of state-contingent capital taxation, as shown by [Chien and Lee \(2005\)](#).

Since the portfolio constraint (23) can be binding only for one of the two households, it follows that  $\gamma_i(s^{t+1}) = 0$  for at least one household. Given this, equations (4) and (7) of the competitive equilibrium can be rewritten as

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \beta \max_{i=1,2} \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} F_K(s^{t+1}) \quad (24)$$

and

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \max_{i=1,2} \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\},$$

where we have substituted for  $r(s^{t+1})$  from (2). Clearly, the two equilibrium conditions (22) and (24) cannot be satisfied by the same allocation  $(\{c_i(s^t)\}_{i \in I}, K)$  if the participation constraint is ever binding, that is, if  $v_i(s^{t+1}) > 0$  for some  $i \in I$  and some  $s^{t+1}|s^t$  with  $\pi(s^{t+1}|s^t) > 0$ . Thus, the constrained efficient allocations cannot be decentralized as competitive equilibria with borrowing constraints that are not too tight.  $\square$

Several points are worth noting. First, this result is in contrast to the one obtained by [Alvarez and Jermann \(2000\)](#), who show that decentralization of the constrained efficient allocations with borrowing constraints that are not too tight is possible in the absence of capital accumulation. Second, a similar result has been shown by [Kehoe and Perri \(2002a\)](#) for a two-country economy with no financial intermediaries.<sup>9</sup> While there are important differences between our setup and theirs, most of our results apply to their framework with minor modifications if we introduce competitive financial intermediaries that operate the production technologies in both countries. This is further discussed in [Section 5.1](#), where our setup is compared extensively to the two-country framework studied by the previous authors.

In what follows, we show that the constrained efficient allocations can be decentralized with borrowing constraints on the Arrow securities that are not too tight if one also imposes a savings constraint on the capital holdings of the intermediary. This is stated in the following proposition, which is the second fundamental theorem of welfare economics for our environment. The proof of this result is relegated to the [Appendix](#).

**PROPOSITION 2.** *Let  $(\{c_i(s^t)\}_{i \in I}, K)$  be a constrained efficient allocation where  $c(s^t) = \sum_i c_i(s^t)$  has high implied interest rates. Further, assume that the intermediary in the decentralized economy is subject to endogenously determined capital accumulation constraints of the form  $k(s^t) \leq B(s^t)$ . Then, the constrained efficient allocations can be decentralized as competitive equilibria with borrowing constraints that are not too tight.*

The proof of this proposition extends the ones in [Alvarez and Jermann \(2000\)](#) and [Kehoe and Perri \(2002a\)](#), where the authors focus on exchange economies, to the presence of production and financial intermediaries that are subject to capital accumulation constraints.

First, we show that  $\{B\}$  can be set so that a constrained efficient allocation that satisfies the planner's capital Euler equation (18) also satisfies the optimality condition of

<sup>9</sup>See also [Seppälä \(1999\)](#) for a similar argument in a related two-sector model.

the intermediary (4) in the competitive equilibrium. Second, the allocations of the planner's problem can be used to construct the dividends  $d$  and share prices  $p$ , as well as the factor prices  $(r, w)$  and the Arrow security prices  $q$ , that satisfy the optimality conditions of the firms and the households. Further, we can iterate on the budget constraints in the competitive equilibrium to obtain the wealth levels  $\{\omega_i\}_{i \in I}$  that support the optimal allocations at every node. In particular, by varying  $\alpha_i$  for  $i \in I$ , we can recover any possible initial wealth distribution. It is then easy to see that the constructed allocations clear the markets and satisfy the transversality condition.

In addition, we first set the borrowing limits  $\{A_i\}_{i \in I}$  equal to  $\{\omega_i\}_{i \in I}$  whenever the participation constraints in the planner's problem are binding and to the natural borrowing limit otherwise. Finally, we can construct the value functions in the competitive equilibrium from the value functions of the planner's problem and redefine the borrowing limits so that they are not too tight for the cases where the participation constraint in the planner's problem is not binding. This way, the constructed allocations with the new borrowing limits are still feasible and optimal.

### 5.1 A two-sector model

In this section we discuss how our previous findings apply to the two-country framework with no financial intermediaries studied by Kehoe and Perri (2002a).

Consider a two-country (sector) framework with no financial intermediaries. In such a framework, each country consists of a large number of identical agents with a production technology that is subject to a country specific shock  $\{z_i\}$ . Let  $F^i(s^t) \equiv f(z_i(s^t), k_i(s^{t-1}), l_i(s^t)) + (1 - \delta)k_i(s^{t-1})$  denote the production of country  $i \in I$  including undepreciated capital. The constrained efficient allocations solve the following problem:

$$\begin{aligned} \max_{\{c_i, k_i\}} \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.} \\ \sum_{i \in I} c_i(s^t) + \sum_{i \in I} k_i(s^t) = \sum_{i \in I} F^i(s^t) \quad \forall s^t \\ \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V(S_i(s^t)) \quad \forall i \text{ and } \forall s^t. \end{aligned}$$

As in our model, optimality requires that equation (17) be satisfied. In addition, it is easy to show that the following condition has to hold for each  $i \in I$  and all  $s^t$ :

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \max_j \frac{u'(c_j(s^{t+1}))}{u'(c_j(s^t))} F_k^j(s^{t+1}) - \frac{v_i(s^{t+1})}{u'(c_i(s^t))} V_k(S_i(s^{t+1})) \right\}. \quad (25)$$

Note that unlike in our setup with only one sector, this condition is not the same for the two agents due to the fact that  $F_k^j(s^{t+1})$  differs between them. Consider now a decentralization of the constrained efficient allocations as competitive equilibria with endogenous debt limits but no capital accumulation constraints and no financial intermediaries, as in Kehoe and Perri (2002a). In this case, the problem of the countries can

be written as

$$\begin{aligned} & \max_{\{c_i, a_i, k_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.} \\ & c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) + k_i(s^t) \leq a_i(s^t) + F^i(s^t) \quad \forall s^t, \\ & k_i(s^t) \geq 0 \text{ and } a_i(s^{t+1}) \geq A_i(s^{t+1}) \quad \forall i \text{ and } \forall s^t. \end{aligned}$$

As in our setup, optimality implies that Arrow security prices are given by

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \left\{ \max_i \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} \quad \forall s^{t+1}|s^t.$$

In addition, the following Euler condition has to hold for each household:<sup>10</sup>

$$1 = \sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} F_k^i(s^{t+1}) \right\} \quad \forall i \text{ and } \forall s^t. \quad (26)$$

As pointed out by [Kehoe and Perri \(2002a\)](#), it is easy to see that equations (25) and (26) cannot be satisfied by the same allocation. Given this, the authors conclude that the constrained efficient allocations in their framework cannot be decentralized as competitive equilibria with endogenous borrowing constraints. In their case the decentralization becomes impossible for two reasons. First, the marginal rates of intertemporal substitution are not aligned between equations (25) and (26). Second, there are autarky effects in (25). In contrast, only the latter effect is present in our framework. The key reason is the absence of financial intermediaries. To see this, suppose that we introduce financial intermediaries that operate the production technology and that purchase capital by issuing Arrow securities in the international market. In this case, the budget constraint of household  $i$  at  $s^t$  is given by

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) \leq a_i(s^t).$$

The intermediaries solve

$$\max_{\{k_i(s^t)\}_{i \in I}} \sum_{i=1,2} \left\{ -k_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) F^i(s^{t+1}) \right\}.$$

It is easy to see that, in the presence of intermediaries, the impossibility of decentralization with borrowing constraints is due only to the autarky effects. While the decentralization with capital accumulation constraints described by [Proposition 2](#) is also possible, the authors study instead a context with no financial intermediaries and they focus on decentralization with governmental default and capital income taxation.

<sup>10</sup>Note that the nonnegativity restriction on capital holdings is not binding in equilibrium.



## 6. CONCLUSIONS

The present paper shows that, in contrast to the findings in exchange economies, the constrained efficient allocations of a model with limited commitment and capital accumulation cannot be decentralized as competitive equilibria with borrowing constraints that are not too tight. Our key result is that, with the introduction of financial intermediaries, this decentralization becomes possible by imposing an upper limit on the intermediary's capital holdings. Moreover, we show that this result can be applied also to a framework with two sectors, which has been studied in the literature.

Our decentralization results can be used to analyze several applied questions where capital accumulation, aggregate uncertainty, and limited commitment are all relevant. As an example, one could study consumption and wealth inequality along the growth path, where capital accumulation can play an important role in determining the incentives to default. Note that in such an environment, our results suggest that there is a negative relationship between the extent of risk sharing and the accumulation of aggregate capital. Ceteris paribus, the higher is the capital growth, the higher are the incentives for default and the tighter are the endogenous borrowing constraints. The optimally chosen capital accumulation constraints thus have to balance this adverse effect with the usual beneficial role of higher aggregate capital. As shown by [Ábrahám and Cárceles-Poveda \(2006\)](#), who study the present model numerically, these capital accumulation effects can be quantitatively important.

## APPENDIX

**PROOF OF PROPOSITION 2.** To prove the proposition, we first note that the capital accumulation constraint  $B(s^t)$  can be set so that a constrained efficient allocation that satisfies the planner's capital Euler equation (22) also satisfies the optimality condition (4) of the intermediary. In particular, when the enforcement constraint in the planner's problem does not bind for any household at period  $t + 1$ , implying that  $v_i(s^{t+1}) = 0$  for  $i = 1, 2$  and all  $s^{t+1}|s^t$ ,  $B(s^t)$  is set to an arbitrary large number so that  $B(s^t) > K(s^t)$ , where  $K(s^t)$  is capital stock in the planner's problem. In this case,  $\psi(s^t) = 0$ . When the enforcement constraint in the planner's problem is binding for either of the two households,  $B(s^t)$  is set to the level of capital that solves the optimal allocation. In this case, equations (22) and (4) imply that the multiplier of the capital accumulation constraint is given by

$$\psi(s^t) = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{i=1,2} \frac{v_i(s^{t+1})}{u'(c_i(s^t))} V_K(S_i(s^{t+1})) \right\}.$$

Note that  $V_K(S_i(s^{t+1})) \geq 0$  for  $i = 1, 2$  due to our assumptions on the production function, which imply that the marginal product of labor is increasing in capital. Given this,  $v_i(s^{t+1}) \geq 0$ , and  $u'(c_i(s^t)) \geq 0$ , it follows that  $\psi(s^t) \geq 0$ .

The factor prices  $w(s^t)$  and  $r(s^t)$  that satisfy the optimality conditions of the firm in the competitive equilibrium can be constructed from the capital allocation of the

planner's problem using equations (1) and (2). Further, the consumption allocations from the planner's problem and equations (20) and (21) can be used to define the prices  $q(s^{t+1}|s^t) = q_p(s^{t+1}|s^t)$  and  $Q(s^{t+1}|s^t) = Q_p(s^{t+1}|s^t)$ . In addition,  $q(s^{t+1}|s^t)$  can be used to define the multiplier  $\gamma_i(s^{t+1})$  so that the asset Euler condition of the agents in equation (7) is satisfied. It is easy to check that the multiplier has the desired properties. In particular, if  $v_i(s^{t+1}) = 0$ , then  $\gamma_i(s^{t+1}) = 0$ . Further, if  $v_i(s^{t+1}) > 0$ , it follows that  $\gamma_i(s^{t+1}) > 0$ . To see this, suppose that  $v_j(s^{t+1}) > 0$  for some  $j = 1, 2$ . Then

$$\beta \pi(s^{t+1}|s^t) \frac{u'(c_j(s^{t+1}))}{u'(c_j(s^t))} < \max_{i=1,2} \left\{ \pi(s^{t+1}|s^t) \beta \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\}$$

and

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \max_{i=1,2} \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} = \beta \pi(s^{t+1}|s^t) \frac{u'(c_j(s^{t+1}))}{u'(c_j(s^t))} + \frac{\gamma_i(s^{t+1})}{u'(c_j(s^t))}.$$

Since the high implied interest rate condition holds, we can then use the budget constraint of the households in the competitive equilibrium to construct the wealth levels  $\omega_i(s^t)$  that support the constrained efficient consumption allocations at every node. To do this, we first construct the profits  $d(s^t)$  from (5), the share price  $p(s^t)$  from (8), and the individual labor incomes from  $w_i(s^t) = w(s^t)\varepsilon_i(s^t)$ . We iterate on the budget constraint of each household to obtain

$$\omega_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \bar{c}_i(s^{t+n}) \quad (27)$$

and

$$\omega_i(s^0) = \sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \bar{c}_i(s^t). \quad (28)$$

Note that we can choose the initial Pareto weights  $\alpha_i$  for  $i = 1, 2$  so that we exactly recover the initial wealth levels in the competitive equilibrium. Concerning the trading limits, if  $v_i(s^{t+1}) = 0$  for agent  $i$ , we set the limits at the natural borrowing constraint, which is given by

$$A_i(s^{t+1}) = - \sum_{n=1}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) [w_i(s^{t+n})\varepsilon_i(s^{t+n}) + \theta_i^0(s^{t+n-1})p(s^{t+n})]$$

and we redefine the limit for these cases later. Further, if  $v_i(s^{t+1}) > 0$ , we set  $A_i(s^{t+1}) = \omega_i(s^{t+1})$ , implying that the borrowing limit is binding when the participation constraint

in the planner's problem is binding. The transversality condition is satisfied, since

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) [\omega_i(s^t) - A_i(s^t)] \\
& \leq \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) \left[ \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \right] \\
& \leq u'(c_i(s^0)) \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) \frac{u'(c_i(s^t))}{u'(c_i(s^0))} \left[ \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n}) \right] \\
& \leq u'(c_i(s^0)) \lim_{t \rightarrow \infty} \sum_{s^t} Q(s^t|s^0) \left[ \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n}) \right] \\
& = 0.
\end{aligned}$$

The first inequality follows from the fact that  $\omega_i(s^t) - A_i(s^t)$  is equal to zero if the participation constraint is binding and is equal to  $\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \geq 0$  otherwise, since  $\omega_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \bar{c}_i(s^{t+n})$ . The second inequality follows from the fact that  $c_i(s^t) \leq \sum_i c_i(s^t)$ . The third inequality follows from the definition of  $Q(s^t|s^0)$  and the fact that  $Q(s^t|s^0) \geq \beta^t \pi(s^t) u'(c_i(s^t)) / u'(c_i(s^0))$  by construction. Finally, the last equality follows from the high implied interest rate condition.

To show that markets clear, we can sum the total asset wealth in (27) and (28), obtaining

$$\begin{aligned}
\sum_i \omega_i(s^t) &= \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i \bar{c}_i(s^{t+n}) = F_K(s^t) K(s^{t-1}) + d(s^t) \\
\sum_i \omega_i(s^0) &= \sum_{t=0}^{\infty} \sum_{s^t} Q(s^t|s^0) \sum_i \bar{c}_i(s^t) = F_K(s^0) K(s^0) + d(s^0).
\end{aligned}$$

Here, we have used the definitions of  $r(s^t)$  and  $p(s^t)$  and the fact that  $\sum_i a_i(s^t) = [r(s^t) + (1 - \delta)] K(s^{t-1})$  and  $\sum_i \theta_i(s^t) = 1$ . In addition, summing the two budget constraints, we have

$$\begin{aligned}
\sum_i c_i(s^t) &= \sum_i \omega_i(s^t) + p(s^t) + w(s^t) - \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) \sum_i \omega_i(s^{t+1}) \\
&= F(s^t) - K(s^t).
\end{aligned}$$

where we have used the definitions of  $p(s^t)$  and  $d(s^t)$  in (8) and (5) and the homogeneity of degree 1 of the production function.

It remains only to redefine the borrowing limits so that they are not too tight. To do this, we construct the autarky values at each node using the allocations of the planner:

$$V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi(s^r) u(f_L(s^r) \varepsilon_i(s^r)).$$

Similarly, we can construct the value function  $W^{ce}(\omega_i(s^t), S_i(s^t))$  and use the two functions to redefine the borrowing constraints for the nodes where the limit is not binding. In particular, we can iterate on the constraint  $A_i(s^t)$  until we find the one that satisfies  $W^{ce}(A_i(s^t), S_i(s^t)) = V^{ce}(S_i(s^t))$ . Since the new set of constraints is (weakly) tighter than before, the new value of  $\omega_i - A_i$  still satisfies the transversality condition. Further, since, these constraints do not bind for any household for whom the participation constraint is not binding in the planner's solution, the allocation derived above with the original constraints is still feasible and optimal.  $\square$

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