Limited Commitment and Hidden Storage

VERY PRELIMINARY AND INCOMPLETE

Árpád Ábrahám* and Sarolta Laczó†

February 15, 2011

Abstract

In the basic model of risk sharing with limited commitment (Kocherlakota, 1996), agents do not have access to any intertemporal storage technology, or, equivalently, all financial transactions are fully observable and contractible. In our model, agents have a private (non-contractible and/or non-observable) storage technology. This does not only affect the value of default, but it also enlarges the set of possible deviations along the equilibrium path. We first study whether and under what conditions the possibility of hidden storage affects the optimal allocation. We find that a necessary condition for it to matter is that income takes more than two values. We then provide an algorithm to solve the model. The main technical difficulty of formulating this problem recursively comes from the fact that the Euler equation determining the optimal storage of agents becomes a constraint. Hence, the temporary Pareto weight (or the life-time utility) of an agent is no longer a sufficient statistic to summarize the individual history, and the state space needs to be further extended. In terms of economic implications, compared to the original model, this extension implies less risk sharing (more consumption dispersion) and a richer dynamics of consumption. In the light of recent research on these models, both results bring them closer to the data.

Keywords: risk sharing, limited commitment, hidden storage, dynamic contracts

JEL codes: E20

*European University Institute, Department of Economics, Villa San Paolo, Via della Piazzuola 43, 50133 Firenze (FI), Italy. Email: arpad.abraham@eui.eu

†University of California, Los Angeles, Department of Economics, Bunche Hall 8369, Los Angeles, CA 90095, United States. Email: slaczo@econ.ucla.edu

1 Introduction

Evidence from both developed and developing countries suggest that households' consumption is partially insured. On the one hand, perfect risk sharing has been rejected by a mounting number of papers (Cochrane, 1991; Mace, 1991; Townsend, 1994, and many others). On the other hand, households are able to mitigate the effects of income shocks on their consumption, and recent evidence suggests that they achieve more insurance than what would be possible by simple self-insurance, i.e. by trading a non-state-contingent bond (Krueger and Perri, 2006, KP hereafter). There is also direct evidence on state-contingent transfers between households (Udry, 1994).

Models that limit access to insurance endogenously while allowing for state-contingent securities have been successful in generating partial insurance. The literature has focused on two imperfections, namely private information (Wang, 1995; Ligon, 1998) and lack of commitment (Kocherlakota (1996); Ligon, Thomas, and Worrall (2002); KP).

This paper extends the literature on risk sharing with limited commitment by introducing a private (non-contractible and/or non-observable) storage technology that agents have access to. We consider two infinitely-lived, risk-averse agents who receive a risky endowment each period. They may make transfers to each other in order to smooth consumption. In addition, they have access to a private storage technology. Transfers between agents are subject to limited commitment, as in Kocherlakota (1996). The storage technology allows agents to transfer consumption from one period to the next, but in a less efficient way than the social planner. Access to hidden storage not only changes the value of autarky, but also enlarges the set of possible deviations along the equilibrium path. To rule out deviations including storage, which is inefficient as it does not earn any return, the Euler equations of both agents become part of the constraint set, in addition to the participation/enforceability constraints.

We first study under what conditions the Euler constraints may bind. We show that, when storage yields zero interest and income may only take two values, Euler constraints never bind in the constrained-optimal allocation. In other words, even if agents can access a private storage technology, the constrained-efficient allocation will be unaffected in this case. However, we also show that if income takes at least three values, the possibility of hidden storage will surely affect the optimal allocation whenever the efficient allocation is characterized by partial insurance and it is not 'too close' to full insurance.

We also study the benchmark case when the rate of return on storage is as high as the rate of time preference. We show that in this case, independently of the income process, the Euler equation binds whenever the optimal allocation does not exhibit full risk sharing.

Thus, hidden storage modifies the optimal allocation. This result implies that if the private technology is not storage but a risk-free non-contingent saving which pays a higher than zero interest rate, then the difference compared to the constrained efficient allocation of the standard model is even bigger.

In order to be able to obtain further characterization and to provide a feasible solution algorithm, we then derive a recursive formulation of the problem with limited commitment and hidden storage. The introduction of the Euler inequalities as constraints means that temporary Pareto weights (Marcet and Marimon, 2009) or promised utilities are no longer sufficient statistics to summarize the past history. Ábrahám and Pavoni (2008) and Werning (2001) show how hidden storage can be accommodated in a recursive manner in dynamic moral hazard models. In this paper, we follow a similar approach. In particular, we propose the following two state variables:[1] past consumption and current ex-ante life-time utility of one of the two agents. It is clear that, under i.i.d. shocks, life-time utilities are sufficient to keep track of the history of the participation constraints, as in the standard model. At the same time, the introduction of past consumption will guarantee that the allocation takes into account the incentives of both agents for using the private storage technology. This is because there is a one-to-one relationship between consumption and the marginal utility of consumption, and the latter is the key for storage incentives.

As we discuss below, this recursive formulation already provides us some analytical insights about consumption dynamics in this environment. In terms of solving this model numerically, we face a couple of challenges. First, compared to the standard model, our model features an additional state variable, past consumption. More importantly, similarly to the dynamic moral hazard environment with hidden savings/storage, the incentive-feasible consumption – life-time utility pairs are endogenous. Moreover, as opposed to the dynamic moral hazard environment with hidden savings/storage, this set cannot be studied in separation from the solution of the constrained-efficient allocation. This is because both agents are subject to the limited commitment problem, hence both agents' life-time utilities show up in the constraint set. Therefore, we propose a solution which directly computes the whole feasible set of utility and consumption pairs in the first step. Then, the frontier of this set will give us the whole set of constrained-optimal allocations.

In terms of economic implications, our model delivers different results compared to the standard model in two dimensions. First, binding Euler constraints further limit risk sharing, therefore our model implies more consumption dispersion. This may bring the model closer to

[1] For expositional simplicity we assume that income shocks are independently and identically distributed across time. Otherwise, current income realizations should also be included in the state space.

the data, since KP have found that the basic model implies too much risk sharing compared to the data. Second, our model generates richer consumption dynamics in two respects. (i) The standard model exhibits amnesia in the sense that the consumption allocation is independent of all the preceding history and is only a function of the current income distribution whenever a participation constraint is binding (Kocherlakota, 1996). In our model, this does not have to be the case. In particular, the allocation is going to be different depending on whether the agents' Euler conditions were binding or not in the previous period. (ii) In the standard model whenever the incomes of the two agents do not change too much, the consumption allocation shows an extreme form of persistence, i.e. it is constant. Again, this is not necessarily the case in our set-up, as the consumption allocation also depends on whether the current and past incentives for storage are the same. Data on household income and consumption do not support neither the very strong amnesia, nor the strong persistence property of the standard model (see Broer, 2009, for an extensive analysis). Hence, these differences are steps in the right direction for this framework to explain consumption dynamics.

The obvious next step of this project is to use the above computational algorithm and study what are the quantitative implications of hidden storage in a relevant economic application, e.g. the transmission of income inequality to consumption inequality, or understanding informal risk sharing in developing countries. This can be very useful both from the positive (we can fit the existing data better) and from the normative (policy implications can be quite different in the presence of private storage) point of view.

The rest of the paper is structured as follows. Section 2 studies whether and under what conditions Euler constraints are binding in the constrained-optimal allocation of the model of risk sharing with limited commitment. Section 3 introduces hidden storage, provides a recursive formulation of this extended model, and discusses some properties of the solution. An algorithm to solve the model is proposed in Section 4.

2 The basic model

We consider an endowment economy with two agents who are infinitely-lived and risk averse. Agents are ex-ante identical in the sense that they have the same preferences and are endowed with the same exogenous random income. Let s_t denote the income state realized today, and s^t the history of income realizations, that is, $s^t = (s_1, s_2, ..., s_t)$. Income realizations are common knowledge. Let $u()$ denote the utility function, that is strictly increasing, strictly concave, and twice continuously differentiable. Assume that the Inada conditions hold.

Suppose that risk sharing is limited by two-sided lack of commitment to insurance contracts, as in Kocherlakota (1996), Ligon, Thomas, and Worrall (2002), and others. In other words, insurance transfers have to be voluntary, or, self-enforcing. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk sharing contracts are sustainable that provide a lifetime utility at least as great as autarky after any history of states for each agent. We assume that the punishment for deviation is exclusion from all risk sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of Abreu (1988).

The constrained-efficient risk sharing contract is the solution to the following maximization problem:

$$\max_{c_i(s^t)} \sum_i \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right), \tag{1}$$

where λ_i is the (initial) Pareto-weight of agent i, β is the the discount factor, $\Pr(s^t)$ is the probability of history s^t occurring, and $c_i\left(s^t\right)$ is the consumption of agent i when history s^t has occurred; subject to the resource constraints,

$$\sum_i c_i\left(s^t\right) \leq \sum_i y_i\left(s_t\right), \forall s^t, \tag{2}$$

and the participation constraints,

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq U_i^{aut}\left(s_t\right), \forall s^t, \forall i, \tag{3}$$

where $\Pr(s^r \mid s^t)$ is the conditional probability of history s^r occurring given that history s^t occurred up to time t, and $U_i^{aut}\left(s_t\right)$ is the expected lifetime utility of agent i when in autarky if state s_t has occurred today.

In the rest of this section, we study whether and under what conditions agents would use a storage technology that yields zero interest rate at the constrained-efficient solution. If Euler constraints may bind, the solution of the basic model is not robust to deviations when hidden storage is available. We first study the case where income may take only two values. Second, we allow for three possible income realizations.

2.1 Two states

Let us consider the case with two possible income realizations. This setting is similar to the "simple model" of Krueger and Perri (2006), KP hereafter. Suppose that each agent is earning $(1+\varepsilon)$ or $(1-\varepsilon)$ with equal probabilities, and $0 < \varepsilon < 1$. Note that ε can be thought of as a measure of income risk, or as a measure of cross-sectional income inequality (as in KP). Given β, the per-period continuation value of autarky when receiving a positive income shock today is

$$U(1+\varepsilon) = \left(1-\frac{\beta}{2}\right) u(1+\varepsilon) + \frac{\beta}{2} u(1-\varepsilon),$$

while sharing risk perfectly would give $u(1)$.

Lemma 1. *(i) $U(1+\varepsilon)$ is strictly increasing in ε at $\varepsilon = 0$, is strictly decreasing in ε as $\varepsilon \to 1$, and is strictly concave. (ii) $U(1+\varepsilon)$ has a unique maximum at ε_1, and $U(1+\varepsilon_1) > u(1)$. (iii) If $\varepsilon \leq \varepsilon_1$, then no insurance transfers are made, and agents stay in autarky. (iv) If $\varepsilon > \varepsilon_1$, then either $U(1+\varepsilon) \leq u(1)$ and perfect risk sharing is self-enforcing, or there exists $0 < \varepsilon^* < \varepsilon_1$ such that $U(1+\varepsilon^*) = U(1+\varepsilon)$.*

Proof. The statements follow from lemma 1 and proposition 2 of KP. □

When ε is such that $U(1+\varepsilon) > u(1)$ and $\varepsilon > \varepsilon_1$ (the case where the participation constraints bind, but positive insurance transfers are made), to find the constrained-efficient sharing rule we have to solve

$$\left(1-\frac{\beta}{2}\right) u(1+\varepsilon^*) + \frac{\beta}{2} u(1-\varepsilon^*) = \left(1-\frac{\beta}{2}\right) u(1+\varepsilon) + \frac{\beta}{2} u(1-\varepsilon) \tag{4}$$

for $0 < \varepsilon^* < \varepsilon$. Note that the smaller ε^* is, the less consumption varies across states, thus the higher ex-ante welfare is. Further, ε_1 maximizes $U(1+\varepsilon)$, that is,

$$\varepsilon_1 = \arg\max_{\varepsilon} \left(1-\frac{\beta}{2}\right) u(1+\varepsilon) + \frac{\beta}{2} u(1-\varepsilon).$$

Differentiating, we have the following lemma:

Lemma 2. *The threshold income risk between partial insurance and autarky, ε_1, is implicitly given by*

$$\left(1-\frac{\beta}{2}\right) u'(1+\varepsilon_1) - \frac{\beta}{2} u'(1-\varepsilon_1) = 0. \tag{5}$$

At the solution each agent consumes $(1+\varepsilon^*)$ or $(1-\varepsilon^*)$ with equal probabilities, and $0 \leq \varepsilon^* \leq \varepsilon_1$. ε^* can be thought of as a measure of consumption risk, or as a measure of cross-sectional consumption inequality (as in KP). Let us study for which values of ε^* the agent would want to use storage to self-insure. Storage yields no interest but may still be useful for precautionary reasons. Note that at the first-best no storage would take place, since there is no aggregate risk (or, the community or the social planner can save and borrow), thus agents' consumption would be perfectly smooth.

The agent with high income today would not want to have a strictly positive stock of savings, when the interest rate equals zero, if and only if

$$u'\left(1+\varepsilon^{*}\right) \geq \beta\left(\frac{1}{2} u'\left(1+\varepsilon^{*}\right)+\frac{1}{2} u'\left(1-\varepsilon^{*}\right)\right). \tag{6}$$

Let ε_1^* denote the level of consumption risk above which the agent would deviate by using hidden storage.

Lemma 3. *The threshold consumption risk above which the Euler constraint (6) binds, ε_1^*, is implicitly given by*

$$\left(1-\frac{\beta}{2}\right) u'\left(1+\varepsilon_1^{*}\right)-\frac{\beta}{2} u'\left(1-\varepsilon_1^{*}\right)=0. \tag{7}$$

Proof. Rearranging (6) and noting that the left hand side of (7) is decreasing in ε_1^* gives the result. □

We are now ready to state our results with respect to what happens when hidden storage is available in the two-states case.

Proposition 1. $\varepsilon_1^* = \varepsilon_1$. *That is, (i) at the solution of the basic model, agents never want to deviate by using hidden storage to better insure; and (ii) in autarky, if $\varepsilon > \varepsilon_1$, agents want to have a positive amount stored given a high income realization, thus storage raises the value of autarky. Therefore, (iii) storage reduces overall insurance, that is, ε^* increases. This implies that ex-ante welfare decreases. Strictly so, whenever partial insurance occurs.*

Proof. Comparing (5) and (7) the results follow. □

Note that, in this environment with no aggregate risk, no storage is optimal at the first best. We have found that agents prefer not to store at the solution of the basic model of risk sharing with limited commitment for any level of income risk. On the other hand, they would store in autarky (for $\varepsilon > \varepsilon_1$, the interesting case), thus the value of the outside option increases. This means that the availability of hidden storage implies less risk sharing at the constrained-efficient solution whenever partial insurance occurs.

2.2 Three states

Let us now consider the case with three income states, while keeping the assumption of constant aggregate income. Suppose that each agent earns $(1+\varepsilon)$, 1, or $(1-\varepsilon)$ with probabilities $\frac{1-\alpha}{2}$, α, and $\frac{1-\alpha}{2}$, respectively, and incomes are perfectly negatively correlated.

Let $U^{aut}(1+\varepsilon \mid \alpha)$ denote the expected lifetime utility in autarky given that the agent has the high endowment today. In mathematical terms,

$$U^{aut}(1+\varepsilon \mid \alpha) = \left(1 - \frac{1+\alpha}{2}\beta\right) u(1+\varepsilon) + \alpha\beta u(1) + \frac{1-\alpha}{2}\beta u(1-\varepsilon).$$

It is easy to see that the following properties from the two-states case still hold:

Lemma 4. *(i) $U^{aut}(1+\varepsilon \mid \alpha)$ is strictly increasing in ε at $\varepsilon = 0$, is strictly decreasing in ε as $\varepsilon \to 1$, and is strictly concave. (ii) $U^{aut}(1+\varepsilon \mid \alpha)$ has a unique maximum at $\varepsilon_1(\alpha)$, and $U^{aut}(1+\varepsilon_1(\alpha) \mid \alpha) > u(1)$. (iii) If $\varepsilon \leq \varepsilon_1(\alpha)$, then no insurance transfers are made, and agents stay in autarky. (iv) If $\varepsilon > \varepsilon_1(\alpha)$, then either $U^{aut}(1+\varepsilon \mid \alpha) \leq u(1)$ and perfect risk sharing is self-enforcing, or partial insurance occurs.*

As in the two-states case, we can simply characterize the level of income risk $\varepsilon_1(\alpha)$ that maximizes $U^{aut}(1+\varepsilon \mid \alpha)$ and below which no insurance transfers are made.

Lemma 5. *Given the probability of the symmetric state and the discount factor, the threshold income risk between partial insurance and autarky, $\varepsilon_1(\alpha)$, is implicitly given by*

$$\left(1 - \frac{1+\alpha}{2}\beta\right) u'(1+\varepsilon_1(\alpha)) - \frac{1-\alpha}{2}\beta u'(1-\varepsilon_1(\alpha)) = 0.$$

The following Euler inequality has to be satisfied for the agent not to want to store secretly when he is in autarky:

$$u'(1+\varepsilon) \geq \beta \left(\frac{1-\alpha}{2}u'(1+\varepsilon) + \alpha u'(1) + \frac{1-\alpha}{2}u'(1-\varepsilon)\right). \tag{8}$$

Lemma 6. *The threshold income risk above which the Euler constraint (8) binds, $\breve{\varepsilon}(\alpha)$, is implicitly given by*

$$\left(1 - \frac{1-\alpha}{2}\beta\right) u'(1+\breve{\varepsilon}(\alpha)) - \alpha\beta u'(1) - \frac{1-\alpha}{2}\beta u'(1-\breve{\varepsilon}(\alpha)) = 0. \tag{9}$$

Proof. Rearranging (8) and noting that the left hand side of (9) is decreasing in $\breve{\varepsilon}(\alpha)$ gives the result. $\square$

The following lemma will be useful to establish whether the Euler constraint may bind at the solution of the basic model of risk sharing with limited commitment.

Lemma 7. *$\breve{\varepsilon}(\alpha) < \varepsilon_1(\alpha)$.*

Proof. It is enough to verify that the Euler inequality (8) is not satisfied at $\varepsilon_1(\alpha)$. To see this, let us rewrite (8) as

$$\left(1-\frac{1+\alpha}{2}\beta\right)u'(1+\varepsilon)+\alpha\beta\left(u'(1+\varepsilon)-u'(1)\right)-\frac{1-\alpha}{2}u'(1-\varepsilon)\geq 0.$$

The second term is negative for any $\varepsilon > 0$ and the other two terms sum to 0 at $\varepsilon_1(\alpha)$, thus the Euler constraint is violated. □

Note that this result differs from what we have found in the two-states case. In particular, agents would use the technology not just for levels of income risk above the level that yields the maximum lifetime utility in autarky, $\varepsilon_1(\alpha)$, but also below it as long as $\varepsilon > \breve{\varepsilon}(\alpha)$.

We now turn to characterizing the solution when partial insurance occurs. Then we will be able to study whether the Euler constraint may bind at the solution. Let $\varepsilon_2(\alpha)$ denote the income risk above which perfect risk sharing is self-enforcing. Its existence is guaranteed by Inada conditions, and it is implicitly defined by $U^{aut}\left(1+\varepsilon_2(\alpha)\mid\alpha\right)=u(1)$. Remember that $\varepsilon_1(\alpha)$ denotes the level of income risk below which agents stay in autarky. We still have to distinguish two cases when $\varepsilon_1(\alpha)<\varepsilon<\varepsilon_2(\alpha)$: (i) when the participation constraints of the symmetric state are not binding at the solution, and (ii) when they are binding.

Lemma 8. *There exists $\tilde{\varepsilon}(\alpha)$ such that $\varepsilon_1(\alpha)<\tilde{\varepsilon}(\alpha)<\varepsilon_2(\alpha)$ and $\forall\varepsilon\in[\tilde{\varepsilon}(\alpha),\varepsilon_2(\alpha))$ the constrained-efficient solution is fully characterized by $\varepsilon^*>0$ that is implicitly given by*

$$\begin{aligned}&\left(1-\frac{1-\alpha}{2}\beta-\frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right)u\left(1+\varepsilon^*\right)+\left(\frac{1-\alpha}{2}\beta+\frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right)u\left(1-\varepsilon^*\right)\\&=\left(1-\frac{1+\alpha}{2}\beta\right)u\left(1+\varepsilon\right)+\alpha\beta u(1)+\frac{1-\alpha}{2}\beta u\left(1-\varepsilon\right),\end{aligned}\tag{10}$$

and the participation constraint of the symmetric state,

$$\begin{aligned}&\left(\frac{1-\alpha}{2}\beta+\frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right)u\left(1+\varepsilon^*\right)+\left(1-\frac{1-\alpha}{2}\beta-\frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right)u\left(1-\varepsilon^*\right)\\&\geq\frac{1-\alpha}{2}\beta u\left(1+\varepsilon\right)+\left(1-(1-\alpha)\beta\right)u(1)+\frac{1-\alpha}{2}\beta u\left(1-\varepsilon\right),\end{aligned}\tag{11}$$

is satisfied.

Proof. Let us first consider what happens when $\varepsilon=\varepsilon_2(\alpha)$. Remember that $\varepsilon_2(\alpha)$ is the threshold between perfect risk sharing and partial insurance. That is, by symmetry, both agents are just indifferent between making the transfer of ε or no transfer when they earn $1+\varepsilon$, the high income. Lowering ε, the constraints of these asymmetric states become binding, thus a smaller transfer $(\varepsilon-\varepsilon^*)$ is made. The optimal interval of the symmetric state is non-degenerate in the neighborhood of $\varepsilon_2(\alpha)$ (see Ligon, Thomas, and Worrall, 2002, proposition

2). Therefore, a transfer of ε^* is self-enforcing in the symmetric state for ε sufficiently close to $\varepsilon_2(\alpha)$. Simple algebra yields the left-hand side of equation (10). It is easy to see that as ε decreases further, and ε^* increases at the same time, (11) becomes binding. That is, the agent with low income yesterday and earning 1 today is not willing to 'reciprocate' as much as ε^*, only a smaller amount. We denote this threshold income risk by $\tilde{\varepsilon}(\alpha)$. □

Finally, $\forall \varepsilon \in (\varepsilon_1(\alpha), \tilde{\varepsilon}(\alpha))$, the constrained-efficient solution is fully characterized by ε^* and χ^*, such that $0 < \chi^* < \varepsilon^* < \varepsilon$, that are determined by the following two equations:

$$\begin{aligned}
&\left(1 - \frac{1-\alpha}{2}\beta\right) u\left(1+\varepsilon^*\right) + \left(\alpha\beta - \frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right) u\left(1+\chi^*\right) \\
&+ \frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2 u\left(1-\chi^*\right) + \frac{1-\alpha}{2}\beta u\left(1-\varepsilon^*\right) \\
&= \left(1 - \frac{1+\alpha}{2}\beta\right) u\left(1+\varepsilon\right) + \alpha\beta u(1) + \frac{1-\alpha}{2}\beta u\left(1-\varepsilon\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{1-\alpha}{2}\beta u\left(1+\varepsilon^*\right) + \frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2 u\left(1+\chi^*\right) \\
&+ \left(1 - (1-\alpha)\beta - \frac{\alpha-\alpha^2}{2(1-\alpha\beta)}\beta^2\right) u\left(1-\chi^*\right) + \frac{1-\alpha}{2}\beta u\left(1-\varepsilon^*\right) \\
&= \frac{1-\alpha}{2}\beta u\left(1+\varepsilon\right) + \left(1-(1-\alpha)\beta\right) u(1) + \frac{1-\alpha}{2}\beta u\left(1-\varepsilon\right).
\end{aligned}$$

Thereby we have completed the characterizations of the solution of the basic model with three income states for all possible patterns of binding participation constraints. We are now ready to state the main result of this section.

Proposition 2. *Assume that $\varepsilon_1(\alpha) < \varepsilon < \varepsilon_2(\alpha)$, that is, partial insurance occurs. Assume further that ε is sufficiently close to $\varepsilon_1(\alpha)$. Then the Euler constraint binds.*

Proof. For ε close to $\varepsilon_1(\alpha)$, χ^* is in a small neighborhood of 0, and ε^* is in a small neighborhood of ε. The result follows from Lemma 7 by continuity. □

It is natural that, if for some level of consumption risk agents would deviate to using the storage technology, then this happens when the solution is close to the autarky, that is, state-contingent insurance transfers are small and agents bear quite a lot of consumption risk.

Having established that Euler constraints may bind when income takes more than two values, in Section 3 we turn to the general problem to solve for the risk sharing contract satisfying both participation and Euler constraints.

2.3 Saving

In this section we discuss the benchmark case when agents have access to an efficient saving technology. In particular, assume that savings earn a return r such that $\beta(1+r) = 1$. Studying this case will also yield insight into what would happen for returns such that $0 < r < \frac{1}{\beta} - 1$.

As above, we only examine whether agents would use the available hidden inter-temporal technology at the constrained-efficient solution of the basic model. We do not make any assumption about the number of income states, except that income may take a finite number of values and the support of the income distribution is bounded. Denote by y^h the highest possible income realization.

Claim 1. *Suppose that partial insurance occurs and the hidden saving technology yields a return r such that $\beta(1+r) = 1$. Then the Euler constraint is binding when an agent receives the highest possible income, y^h.*

Proof. Let c^h denote the consumption of the agent receiving income y^h. If partial insurance occurs, as opposed to full insurance, then it must be that there exists some state $\tilde{s}$ where the agent consumes $c(\tilde{s}) < c^h$. Then

$$u'(c^h) < \sum_s \pi(s) u'(c(s)),$$

that is, the Euler constraint is violated. □

This result implies that, if the private technology is not storage but a risk-free non-contingent saving which pays the equilibrium interest rate of the efficient allocation, then the difference compared to the constrained efficient allocation is even bigger. More generally, this result suggests that if the private storage technology pays a higher than zero interest rate, then the Euler constraints are more likely to bind, further limiting risk sharing.

3 The model with hidden storage

We now add Euler constraints, or, *no-storage* constraints, to the problem given by the objective function (1) and the constraints (2) and (3). In mathematical terms, we require the following inequalities to hold:

$$u'\left(c_i\left(s^t\right)\right) \geq \beta \sum_{s^{t+1}} \Pr\left(s^{t+1} \mid s^t\right) u'\left(c_i\left(s^{t+1}\right)\right), \forall s^t, \forall i. \tag{12}$$

Note that both the participation constraints (3) and the Euler constraints (12) involve future decision variables, and there are an infinite number of both types of constraints.

3.1 Recursive form

We consider an ex-ante formulation of the problem of agent 1, using life-time utility and the consumption of the second agent as state variables. We normalize the total endowment to 1. Remember that we have assumed that aggregate consumption can be perfectly smoothed over time.

Let c denote the consumption of agent 2 today and u denote the expected lifetime utility promised to agent 2 from tomorrow onwards.[2] c and u are the state variables. The controls are denoted c^s and w^s, where the upper index s refers to the state of the world tomorrow, and $s \in \{1, 2, ..., S\}$, where S is the number of income states. c^s is the consumption of agent 2 in state s tomorrow, and w^s is his expected lifetime utility from the period after tomorrow onwards when tomorrow's state is s. The probability of state s occurring is denoted π^s. Let $V^{aut}()$ denote the value function of autarky with storage that can be solved for by standard value function iteration. Finally, denote by $V()$ the value function of agent 1. We can then write the problem of agent 1 in the following recursive form:

$$V(u, c) = \max_{\{(w^s, c^s) \in \Lambda\}_{s=1}^{S}} \sum_{s=1}^{S} \pi^s \left(u(1 - c^s) + \beta V(w^s, c^s) \right) \tag{13}$$

subject to

$$u = \sum_{s=1}^{S} \pi^s \left(u(c^s) + \beta w^s \right), \tag{14}$$

$$u(1 - c^s) + \beta V(w^s, c^s) \geq V^{aut}(1 - y^s), \ \forall s, \tag{15}$$

$$u(c^s) + \beta w^s \geq V^{aut}(y^s), \ \forall s, \tag{16}$$

$$\beta \sum_{s=1}^{S} \pi^s u'(1 - c^s) \leq u' \left(1 - c\right), \tag{17}$$

and

$$\beta \sum_{s=1}^{S} \pi^s u'(c^s) \leq u'(c), \tag{18}$$

where Λ denotes all incentive-feasible combinations of the state variables, c and u.

In words, we maximize the expected lifetime utility of agent 1 from tomorrow onwards, subject to the promise keeping constraint (14), which is in terms of the expected lifetime utility of agent 2 from tomorrow onwards; the participation constraints of agents 1 and 2, (15) and (16), respectively; and the Euler constraints of agents 1 and 2, (17) and (18), respectively.

[2]Another interpretation is that c is past consumption and u is life-time utility from the current period onward before observing the shock in the current period.

Note first that, using similar arguments as in Ábrahám and Pavoni (2008), we can show that this recursive formulation is indeed equivalent with the sequential formulation introduced before. Intuitively, it is clear that, under i.i.d. shocks, life-time utilities are sufficient to keep track of the history of participation constraints, as in the standard model. At the same time, the introduction of past consumption will guarantee that the allocation takes into account the incentives of both agents for using the private storage technology. This is because there is a one-to-one relationship between consumption and the marginal utility of consumption, and the latter is the key for storage incentives.

Remember that Λ denotes all the incentive-feasible combinations of the state variables, consumption and life-time utility of agent 2. This set is endogenous and is given by

$$\Lambda = \left\{ \begin{array}{c} (u,c): \\ \exists \{w^s, c^s\}_{s=1}^S \in \Lambda \text{ s.t.} \\ u = \sum_{s=1}^S \pi^s \left(u(c^s) + \beta w_2^s\right) \\ u(c^s) + \beta w^s \geq V^{aut}(y^s),\ \forall s \\ u(1-c^s) + \beta V(w^s, c^s) \geq V^{aut}(y^{S-s+1}),\ \forall s \\ u'(c) \geq \beta \sum_{s=1}^S \pi^s u'(c^s) \\ u'(1-c) \geq \beta \sum_{s=1}^S \pi^s u'(1-c^s) \end{array} \right\}. \tag{19}$$

Note that, through the participation constraint of agent 1, the value function $V()$ is part of the constraint set. This property makes the direct execution of the recursive dynamic programming problem given by equations (13) to (19) difficult.

3.2 Characterization

Let us denote by γ_1^s and γ_2^s the Lagrange multipliers of the participation constraints and by λ_1 and λ_2 the multipliers of the Euler constraints. Finally, μ is the Lagrange multiplier of the promise keeping constraint. It is also the current temporary Pareto weight. Taking first-order conditions we get

$$-\frac{\partial V_s(w^s, c^s)}{\partial w^s}(1+\gamma_2^s) = \mu + \gamma_1^s \tag{20}$$

and

$$\frac{\partial V(w^s, c^s)}{\partial c^s}(1+\gamma_2^s) - \lambda_1 u''(c^s) + \lambda_2 u''(1-c^s) = 0 \tag{21}$$

The envelope conditions yields

$$\frac{\partial V_i(u,c)}{\partial c} = \mu u'(c) - u'(1-c) + \lambda_1 u''(c) - \lambda_2 u''(1-c) \tag{22}$$

and

$$\frac{\partial V_i(u,c)}{\partial u} = \mu. \tag{23}$$

Combining (20) and (23) we get

$$\mu^s = \frac{\mu + \gamma_1^s}{1 + \gamma_2^s}. \tag{24}$$

That is, if neither participation constraint binds, we have that $\mu^s = \mu$. If the one for agent 1 binds, we get that $\mu^s > \mu$. If the one for agent 2 binds, we get that $\mu^s < \mu$, just as in the case without storage.

Combining (21) and (22) we get

$$(1 + \gamma_2^s)\left(\mu^s u'(c^s) - u'(1 - c^s) + \lambda_1^s u''(c) - \lambda_2^s u''(1 - c)\right) - \lambda_1 u''(c^s) + \lambda_2 u''(1 - c^s) = 0.$$

Rewriting yields

$$(1 + \gamma_2^s)\left(\mu^s u'(c^s) - u'(1 - c^s)\right) + u''(c^s)((1 + \gamma_2^s)\lambda_1^s - \lambda_1) - u''(1 - c^s)((1 + \gamma_2^s)\lambda_2^s - \lambda_2) = 0$$

This condition is more difficult to interpret. If $\lambda_1 = \lambda_2 = \lambda_1^s = \lambda_2^s = 0$, that is, Euler constraints are not binding today or tomorrow, then we have $\mu^s = \frac{u'(1-c^s)}{u'(c^s)}$, so we are back to the basic model, as we should be. Otherwise

$$\mu^s = \frac{u'(1 - c^s) - u''(c^s)\left(\lambda_1^s - \frac{\lambda_1}{1+\gamma_2^s}\right) + u''(1 - c^s)\left(\lambda_2^s - \frac{\lambda_2}{1+\gamma_2^s}\right)}{u'(c^s)}. \tag{25}$$

We first consider the cases where only one Euler constraint binds.

- Assume that $\lambda_1 > 0$ and the rest of the λ's are zero. Then c^s needs to increase, which makes sense, because this way we can decrease the incentive of agent 1 to store in the current period.

- Assume that $\lambda_1^s > 0$ and the rest of the λ's are zero. Then c^s needs to decrease, which makes sense, because this way we can decrease the incentive of agent 1 to store in the next period.

- Assume that $\lambda_2 > 0$ and the rest of the λ's are zero. Then c^s needs to decrease, which makes sense, because this way we can decrease the incentive of agent 2 to store in the current period.

- Assume that $\lambda_2^s > 0$ and the rest of the λ's are zero. Then c^s needs to increase, which makes sense, because this way we can decrease the incentive of agent 2 to store in the next period.

One key property of the standard model is that whenever either of the participation constraints is binding ($\gamma_1^s > 0$ or $\gamma_2^s > 0$), then the resulting allocation is independent of the preceding history. In our formulation, this implies that μ^s is only a function of s and not the state variables. This is often called the amnesia property (Kocherlakota, 1996), and typically data do not support this pattern.[3] From equation (25) it is obvious that the amnesia property does not necessarily hold in our model. The consumption allocation does not only depend on variables indexed by s, but also on λ_1 and λ_2. In particular, the allocation of consumption will be different depending on whether either of the agents had binding incentives to use storage in the previous period, and if yes, which of them had these incentives.

Another property of the standard model is that whenever neither participation constraint is binding ($\gamma_1^s = \gamma_2^s = 0$), then we have that the consumption allocation is constant and hence exhibits an extreme form of persistence. This can be seen easily. Given $\lambda_1 = \lambda_2 = \lambda_1^s = \lambda_2^s = 0$, the consumption allocation is only a function of μ^s, and (24) implies $\mu^s = \mu$. This implies that for income changes which do not trigger a binding participation constraint, we do not see any change in the consumption allocation. It is again not easy to find evidence for this pattern in the data. Notice, however, that in our model, even if $\mu^s = \mu$, (25) does not imply that consumptions are the same today as last period. They will only be the same if λ_1 and λ_2 in the current period are the same as they were in the previous period. This implies that consumption does not necessary remain constant, even if the participation constraints do not bind in the current period.

These two observations make the consumption dynamics in the current model much richer than in the standard model, and they bring the implications of this model closer to the data. It is also important to note that, for developed economies, once labor supply decisions and demographics are appropriately accounted for, the Euler equation cannot be rejected in micro data, at least in its inequality form (see Attanasio, 1999, for a comprehensive review of the literature). Since in our model the Euler inequality has to be satisfied by construction, we bring limited commitment models in line with this observation as well.

4 Solving for the Optimal Allocation

As we pointed out above, since the feasibility set, Λ, is endogenous (hence not known ex ante) and involves the value function itself as well, it is difficult to solve directly the dynamic program above. For this reason, we propose a different procedure. This procedure takes advantage of the symmetric nature of the problem and aims at computing the whole feasibility

[3] See for example Kinnan (2010) who rejects the amnesia property using data from Thai villages.

set in terms of life-time utilities and consumptions of both agents. More precisely, we look for feasible triples of (u_1, u_2, c), where c is the consumption of agent 2 today, as before, and u_1 and u_2 are the expected lifetime utilities of agent 1 and 2, respectively, from tomorrow onwards. The feasible set, that we denote B, is given by the following conditions:

$$B = \left\{ \begin{array}{c} (u_1, u_2, c) : \\ \exists \{w_1^s, w_2^s, c^s\}_{s=1}^{S} \in B \text{ s.t.} \\ (w_2^s, w_1^s, 1-c^s) \in B, \ \forall s \\ u_2 = \sum_{s=1}^{S} \pi^s \left(u(c^s) + \beta w_2^s\right) \\ u_1 = \sum_{s=1}^{S} \pi^s \left(u(1-c^s) + \beta w_1^s\right) \\ u(c^s) + \beta w_2^s \geq V^{aut}(y^s), \ \forall s \\ u(1-c^s) + \beta w_1^s \geq V^{aut}(y^{S-s+1}), \ \forall s \\ u'(c) \geq \beta \sum_{s=1}^{S} \pi^s u'(c^s) \\ u'(1-c) \geq \beta \sum_{s=1}^{S} \pi^s u'(1-c^s) \end{array} \right\} = T(B).$$

Hence, we can obtain B as the fixed point of the operator $T(\cdot)$. Moreover, if we start from a large set B^0, the iterative procedure $B^{n+1} = T(B^n)$ will converge to the fixed point B under general conditions.

After the set B is computed, we can find the value function and the optimal allocation defined in (13) to (19) as

$$V(u, c) = \max \{u_1 : (u_1, u, c) \in B\}.$$

That is, the value function is the maximum incentive feasible life-time utility agent 1 can achieve, given that the other agent's past consumption is c and her life-time utility is u. The optimal allocation is then given by the $\{w_2^s, c^s\}_{s=1}^{S}$ from the definition of B, which makes this tuple feasible.

Note also that in the initial period there cannot be a binding storage condition from the previous period which constrains the optimal allocation. For this reason, in this initial period there is only one state variable, the life-time utility that has to be delivered to the other agent. Nevertheless, given $V(u, c)$, it is easy to calculate this initial value and the optimal allocation in this initial period as

$$V^0(u) = \max_{c:(V(u,c),u,c) \in B} V(u, c).$$

Consequently, $c_0 = \arg\max_{c:(V(u,c),u,c)\in B} V(u, c)$, and the optimal allocation in the initial period can be computed accordingly.

References

Ábrahám, Á. and N. Pavoni (2008). Efficient Allocations with Moral Hazard and Hidden Borrowing and Lending: A Recursive Formulation. *Review of Economic Dynamics 11*(4), 781 – 803.

Abreu, D. (1988). On the Theory of Infinitely Repeated Games with Discounting. *Econometrica 56*(2), 383–396.

Attanasio, O. P. (1999). Chapter 11 Consumption. Volume 1, Part 2 of *Handbook of Macroeconomics*, pp. 741 – 812. Elsevier.

Broer, T. (2009). Stationary Equilibrium Distributions in Economies with Limited Commitment. Mimeo.

Cochrane, J. H. (1991). A Simple Test of Consumption Insurance. *Journal of Political Economy 99*(5), 957–976.

Kinnan, C. (2010). Distinguishing Barriers to Insurance in Thai Villages. Mimeo.

Kocherlakota, N. R. (1996). Implications of Efficient Risk Sharing without Commitment. *Review of Economic Studies 63*(4), 595–609.

Krueger, D. and F. Perri (2006). Does Income Inequality Lead to Consumption Inequality? Evidence and Theory. *Review of Economic Studies 73*(1), 163–193.

Ligon, E. (1998). Risk Sharing and Information in Village Economies. *The Review of Economic Studies 65*(4), 847–864.

Ligon, E., J. P. Thomas, and T. Worrall (2002). Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies. *Review of Economic Studies 69*(1), 209–244.

Mace, B. J. (1991). Full Insurance in the Presence of Aggregate Uncertainty. *Journal of Political Economy 99*(5), 928–956.

Marcet, A. and R. Marimon (2009). Recursive Contracts. Mimeo.

Townsend, R. M. (1994). Risk and Insurance in Village India. *Econometrica 62*(3), 539–591.

Udry, C. (1994). Risk and insurance in a rural credit market: An empirical investigation in northern Nigeria. *Review of Economic Studies 61*(3), 495–526.

Wang, C. (1995). Dynamic Insurance with Private Information and Balanced Budgets. *Review of Economic Studies 62*(4), 577–595.

Werning, I. (2001). Repeated Moral Hazard with Unmonitored Wealth: A Recursive First-Order Approach. Mimeo.